

Controlling Chaos

Edward Ott,^{(a),(b)} Celso Grebogi,^(a) and James A. Yorke^(c)

University of Maryland, College Park, Maryland 20742

(Received 22 December 1989)

It is shown that one can convert a chaotic attractor to any one of a large number of possible attracting time-periodic motions by making only *small* time-dependent perturbations of an available system parameter. The method utilizes delay coordinate embedding, and so is applicable to experimental situations in which *a priori* analytical knowledge of the system dynamics is not available. Important issues include the length of the chaotic transient preceding the periodic motion, and the effect of noise. These are illustrated with a numerical example.

PACS numbers: 05.45.+b

The presence of chaos in physical systems has been extensively demonstrated and is very common. In practice, however, it is often desired that chaos be avoided and/or that the system performance be improved or changed in some way. Given a chaotic attractor, one approach might be to make some large and possibly costly alteration in the system which completely changes its dynamics in such a way as to achieve the desired behavior. Here we assume that this avenue is not available. Thus, we address the following question: Given a chaotic attractor, how can one obtain improved performance and a desired attracting time-periodic motion by making only *small* time-dependent perturbations in an *accessible* system parameter?

The key observation is that a chaotic attractor typically has embedded within it an infinite number of unstable periodic orbits.¹ Since we wish to make only small perturbations to the system, we do not envision creating new orbits with very different properties from the existing ones. Thus, we seek to exploit the already existing unstable periodic orbits. Our approach is as follows: We first determine some of the unstable low-period periodic orbits that are embedded in the chaotic attractor. We then examine these orbits and choose one which yields improved system performance. Finally, we tailor our small time-dependent parameter perturbations so as to stabilize this already existing orbit. In this Letter we describe how this can be done, and we illustrate the method with a numerical example. The method is very general and should be capable of yielding greatly improved performance in a wide variety of situations.

It is interesting to note that if the situation is such that the suggested method is practical, then the presence of chaos can be a great advantage. The point is that any one of a number of different orbits can be stabilized, and the choice can be made to achieve the best system performance among those orbits. If, on the other hand, the attractor is not chaotic but is, say, periodic, then small parameter perturbations can only change the orbit slightly. Basically we are then stuck with whatever system performance the stable periodic orbit gives, and we have no option for substantial improvement, short of

making large alterations in the system.

Furthermore, one may want a system to be used for different purposes or under different conditions at different times. Thus, depending on the use, different requirements are made of the system. If the system is chaotic, this type of multiple-use situation might be accommodated without alteration of the gross system configuration. In particular, depending on the use desired, the system behavior could be changed by switching the temporal programming of the small parameter perturbations to stabilize different orbits. In contrast, in the absence of chaos, completely separate systems might be required for each use. Thus, when designing a system intended for multiple uses, purposely building chaotic dynamics into the system may allow for the desired flexibility. Such multipurpose flexibility is essential to higher life forms, and we, therefore, speculate that chaos may be a necessary ingredient in their regulation by the brain.

To simplify the analysis we consider continuous-time dynamical systems which are *three dimensional* and depend on one system parameter which we denote p [for example, $dx/dt = F(x, p)$, where x is three dimensional]. We assume that the parameter p is available for external adjustment, and we wish to temporally program our adjustments of p so as to achieve improved performance. We emphasize that our restriction to a three-dimensional system is mainly for ease of presentation, and that the case of higher-dimensional (including infinite-dimensional) systems can be treated by similar methods.²

We imagine that the dynamical equations describing the system are not known, but that experimental time series of some scalar-dependent variable $z(t)$ can be measured. Using delay coordinates^{3,4} with delay T one can form a delay-coordinate vector,

$$\mathbf{X}(t) = [z(t), z(t-T), z(t-2T), \dots, z(t-MT)].$$

We are interested in periodic orbits and their stability properties, and we shall use \mathbf{X} to obtain a surface of section for this purpose. In the surface of section, a continuous-time-periodic orbit appears as a discrete-time orbit cycling through a finite set of points. We require the dynamical behavior of the surface of section map in

neighborhoods of these points in order to study the stability of the periodic orbits. To embed a small neighborhood of a point from \mathbf{x} into \mathbf{X} , we typically only require as many dimensions as there are coordinates of the point. Thus, for our purposes, $M=D-1$ is generally sufficient. (This is in contrast with³ $M+1=2D+1$, typically required for global embedding of the original phase space in the delay-coordinate space.) Hence, for the case considered ($D=3$), our surface of section is two dimensional.

We suppose that the parameter p can be varied in a small range about some nominal value p_0 . Henceforth, without loss of generality, we set $p_0 \equiv 0$. Let the range in which we are allowed to vary p be $p_* > p > -p_*$.

Using an experimental surface of section for the embedding vector \mathbf{X} , we imagine that we obtain many experimental points in the surface of section for $p=0$. We denote these points $\xi_1, \xi_2, \xi_3, \dots, \xi_k$, where ξ_n denotes the coordinates in the surface of section at the n th piercing of the surface of section by the orbit $\mathbf{X}(t)$. For example, a common choice of the surface of section would be $z(t-MT)$ equals a constant, and $\xi_n = [z(t_n), \dots, z(t_n - (M-1)T)]$, where $t=t_n$ denotes the time at the n th piercing. From such experimentally determined sequences it has been demonstrated that a large number of distinct unstable periodic orbits on a chaotic attractor can be determined.^{5,6} We then examine these unstable periodic orbits and select the one which gives the best performance. Again using an experimentally determined sequence, we obtain the stability properties of the chosen periodic orbit (cf. Refs. 5 and 6 for discussion of how this can be done and for descriptions of its implementation in concrete experimental cases). For the purposes of simplicity, let us assume in what follows that this orbit is a fixed point of the surface of section map (i.e., period one; the case of higher period is a straightforward extension). Let λ_s and λ_u be the experimentally determined stable and unstable eigenvalues of the surface of section map at the chosen fixed point of the map ($|\lambda_u| > 1 > |\lambda_s|$). Let \mathbf{e}_s and \mathbf{e}_u be the experimentally determined unit vectors in the stable and unstable directions. Let $\xi = \xi_F \equiv 0$ be the desired fixed point. We then change p slightly from $p=0$ to some other value $p=\bar{p}$. The fixed-point coordinates in the experimental surface of section will shift from 0 to some nearby point $\xi_F(\bar{p})$ and we determine this new position. For small \bar{p} we approximate $\mathbf{g} \equiv \partial \xi_F(p) / \partial p |_{p=0} \cong \bar{p}^{-1} \xi_F(\bar{p})$, which allows an experimental determination of the vector \mathbf{g} .

Thus, in the surface of section, near $\xi=0$, we can use a linear approximation for the map, $\xi_{n+1} - \xi_F(p) \cong \mathbf{M} \cdot [\xi_n - \xi_F(p)]$, where \mathbf{M} is a 2×2 matrix. Using $\xi_F(p) \cong p\mathbf{g}$ we have

$$\xi_{n+1} \cong p_n \mathbf{g} + [\lambda_u \mathbf{e}_u \mathbf{f}_u + \lambda_s \mathbf{e}_s \mathbf{f}_s] \cdot [\xi_n - p_n \mathbf{g}]. \quad (1)$$

[In the linearization (1), we have considered p_n to be small and of the same order as ξ_n .] We emphasize that

\mathbf{g} , \mathbf{e}_u , \mathbf{e}_s , λ_u , and λ_s are all experimentally accessible by the embedding technique just discussed. In (1) \mathbf{f}_u and \mathbf{f}_s are contravariant basis vectors defined by $\mathbf{f}_u \cdot \mathbf{e}_s = \mathbf{f}_s \cdot \mathbf{e}_u = 1$, $\mathbf{f}_s \cdot \mathbf{e}_u = \mathbf{f}_u \cdot \mathbf{e}_s = 0$. Note that we have written the location of the fixed point as $p_n \mathbf{g}$ because we imagine that we adjust p to a new value p_n after each piercing of the surface of section. That is, we observe ξ_n and then adjust p to the value p_n . Thus p_n depends on ξ_n . Further, we only envision making this adjustment when the orbit falls near the desired fixed point for $p=0$.

Assume that ξ_n falls near the desired fixed point at $\xi=0$ so that (1) applies. We then attempt to pick p_n so that ξ_{n+1} falls on the stable manifold of $\xi=0$. That is, we choose p_n so that $\mathbf{f}_u \cdot \xi_{n+1} = 0$. If ξ_{n+1} falls on the stable manifold of $\xi=0$, we can then set the parameter perturbations to zero, and the orbit for subsequent time will approach the fixed point at the geometrical rate λ_s . Thus, for sufficiently small ξ_n , we can dot (1) with \mathbf{f}_u to obtain

$$p_n = \lambda_u (\lambda_u - 1)^{-1} (\xi_n \cdot \mathbf{f}_u) / (\mathbf{g} \cdot \mathbf{f}_u), \quad (2)$$

which we use when the magnitude of the right-hand side of (2) is less than p_* . When it is greater than p_* , we set $p_n=0$. We assume in (2) that the generic condition $\mathbf{g} \cdot \mathbf{f}_u \neq 0$ is satisfied. Thus, the parameter perturbations are activated (i.e., $p_n \neq 0$) only if ξ_n falls in a narrow strip $|\xi_n^u| < \xi_*$, where $\xi_n^u = \mathbf{f}_u \cdot \xi_n$, and from (2) $\xi_* = p_* | (1 - \lambda_u^{-1}) \mathbf{g} \cdot \mathbf{f}_u |$. Thus, for small p_* , a typical initial condition will execute a chaotic orbit, unchanged from the uncontrolled case, until ξ_n falls in the strip. Even then, because of nonlinearity not included in (1), the control may not be able to bring the orbit to the fixed point. In this case the orbit will leave the strip and continue to wander chaotically as if there was no control. Since the orbit on the uncontrolled chaotic attractor is ergodic, at some time it will eventually satisfy $|\xi_n^u| < \xi_*$ and also be sufficiently close to the desired fixed point that attraction to $\xi=0$ is achieved. [In rare cases applying Eq. (2) when the trajectory enters the strip, but is still far from 0, may result in stabilizing the wrong periodic orbit which visits the strip.]

Thus, we create a stable orbit, but, for a typical initial condition, it is preceded in time by a chaotic transient in which the orbit is similar to orbits on the uncontrolled chaotic attractor. The length τ of such a chaotic transient depends sensitively on the initial condition, and, for randomly chosen initial conditions, has an exponential probability distribution⁷ $P(\tau) \sim \exp[-(\tau/\langle\tau\rangle)]$ for large τ . The average length of the chaotic transient $\langle\tau\rangle$ increases with decreasing p_* and follows a power-law relation⁷ for small p_* , $\langle\tau\rangle \sim p_*^{-\gamma}$.

We will now derive a formula for the exponent γ . Dotted the linearized map for ξ_{n+1} , Eq. (1), with \mathbf{f}_u , we obtain $\xi_{n+1}^u \cong 0$. In obtaining this result from (1) we have substituted p_n appropriate for $|\xi_n^u| < \xi_*$. We note that the result $\xi_{n+1}^u \cong 0$ is a linearization, and typically

has a lowest-order nonlinear correction that is quadratic. In particular, $\xi_n^s = \mathbf{f}_s \cdot \xi_n$ is not restricted by $|\xi_n^u| < \xi_*$, and thus may not be small when the condition $|\xi_n^u| < \xi_*$ is satisfied. Hence the correction quadratic in ξ_n^s is most significant. Including such a correction we have $\xi_{n+1}^u \cong \kappa(\xi_n^s)^2$, where κ is a constant. Thus, if $|\kappa|(\xi_n^s)^2 > \xi_*$, then $|\xi_{n+1}^u| > \xi_*$, and attraction to $\xi = 0$ is not achieved, even though $|\xi_n^u| < \xi_*$. Attraction to $\xi = 0$ is achieved when the orbit falls in the small parallelogram P_c given by $|\xi_n^u| < \xi_*$, $|\xi_n^s| < (\xi_*/|\kappa|)^{1/2}$. For very small ξ_* , an initial condition will bounce around on the set comprising the uncontrolled chaotic attractor for a long time before it falls in the parallelogram P_c . At any given iterate the probability of falling in P_c is $\mu(P_c)$, the measure of the uncontrolled attractor contained in P_c . Thus, $\langle \tau \rangle^{-1} = \mu(P_c)$. The scaling of $\mu(P_c)$ with ξ_* is

$$\mu(P_c) \sim (\xi_*)^{d_u} [(\xi_*/|\kappa|)^{1/2}]^{d_s} \sim \xi_*^{d_u + (1/2)d_s},$$

where d_u and d_s are the partial pointwise dimensions for the uncontrolled chaotic attractor at $\xi = 0$ in the unstable direction and the stable direction, respectively. Thus, $\mu(P_c) = \xi_*^\gamma$, where $\gamma = d_u + d_s/2$. Since we assume the attractor to be effectively smooth in the unstable direction, $d_u = 1$. The partial pointwise dimension in the stable direction is given in terms of the eigenvalues⁷ at $\xi = 0$, $d_s = \ln|\lambda_u|/\ln|\lambda_s|^{-1}$. Thus,

$$\gamma = 1 + \frac{1}{2} \ln|\lambda_u|/\ln|\lambda_s|^{-1}. \tag{3}$$

To study the effect of noise we add a term $\epsilon \delta_n$ to the right-hand side of the linearized equations for ξ_{n+1} , Eq. (1), where δ_n is a random variable and ϵ is a small parameter specifying the intensity of the noise. The quantities δ_n are taken to have zero mean ($\langle \delta_n \rangle = 0$), be independent ($\langle \delta_n \delta_m \rangle = 0$ for $m \neq n$), and have a probability density independent of n . Dotting (1) with noise included with \mathbf{f}_u we obtain $\xi_{n+1}^u = \epsilon \delta_n^u$, where $\delta_n^u \equiv \mathbf{f}_u \cdot \delta_n$. Thus, if the noise is bounded, $|\delta_n^u| < \delta_{\max}$, then the stability of $\xi = 0$ will not be affected by the noise if the bound is small enough, $\epsilon \delta_{\max} < \xi_*$. If this condition is not satisfied, then the noise can kick an orbit which is initially in the parallelogram P_c into the region outside P_c . We are particularly interested in the case where such kickouts are caused by low-probability tails on the probability density and are thus rare. (If they are frequent, then our procedure is ineffective.) In such a case the average time to be kicked out $\langle \tau' \rangle$ will be long. Thus, an orbit will typically alternate between epochs of chaotic motion of average duration $\langle \tau \rangle$ in which it is far from $\xi = 0$, and epochs of average length $\langle \tau' \rangle$ in which the orbit lies in the parallelogram P_c . For small enough noise the orbit spends most of its time in P_c , $\langle \tau' \rangle \gg \langle \tau \rangle$, and one might then regard the procedure as being effective.

We now consider a specific numerical example. Our purpose is to illustrate and test our analyses of the average time to achieve control and the effect of noise. To do

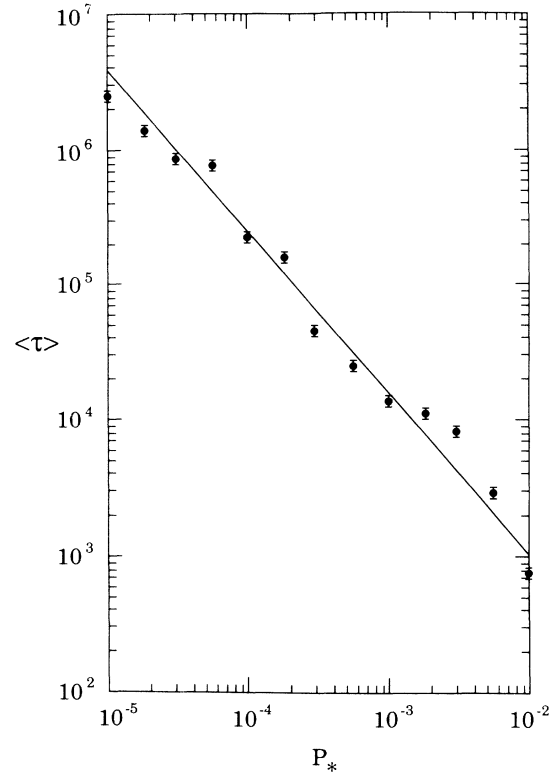


FIG. 1. $\langle \tau \rangle$ vs p_* . Points were computed using 128 randomly selected initial conditions. $A_0 = 1.4$.

this we shall utilize the Henon map, $x_{n+1} = A - x_n^2 + B y_n$, $y_{n+1} = x_n$, where we take $B = 0.3$. We assume that the quantity A can be varied by a small amount about some value A_0 . Accordingly, we write A as $A = A_0 + p$, where p is the control parameter. For the values of A_0 which we investigate, the attractor for the map is chaotic and contains an unstable period-one (fixed-point) orbit. The coordinates (x_F, y_F) of the fixed point which is in the attractor for $p = 0$ along with the associated parameters and vectors appearing in Eq. (1) may be explicitly calculated. The quantity ξ_n appearing in (1) is $\xi_n = (x_n - x_F)\mathbf{x}_0 + (y_n - y_F)\mathbf{y}_0$. To test our prediction for the dependence of $\langle \tau \rangle$, the average time to approach $\xi = 0$, on the maximum allowed size of the parameter perturbation p_* , we proceed as follows. We iterate the map with $p = 0$ using a large number of randomly chosen initial conditions until all these initial conditions are distributed over the attractor (500 iterates were typically used). We then turn on the parameter perturbations and determine for each orbit how many further iterates τ are necessary before the orbit falls within a circle of radius $\frac{1}{2} \xi_*$ centered at the fixed point. We then calculate the average of these times. We do this for many different values of p_* and plot the results as a function of p_* . This is shown on the log-log plot in Fig. 1 along with the theoretical straight line of slope given by the exponent (3). We see that the agreement is

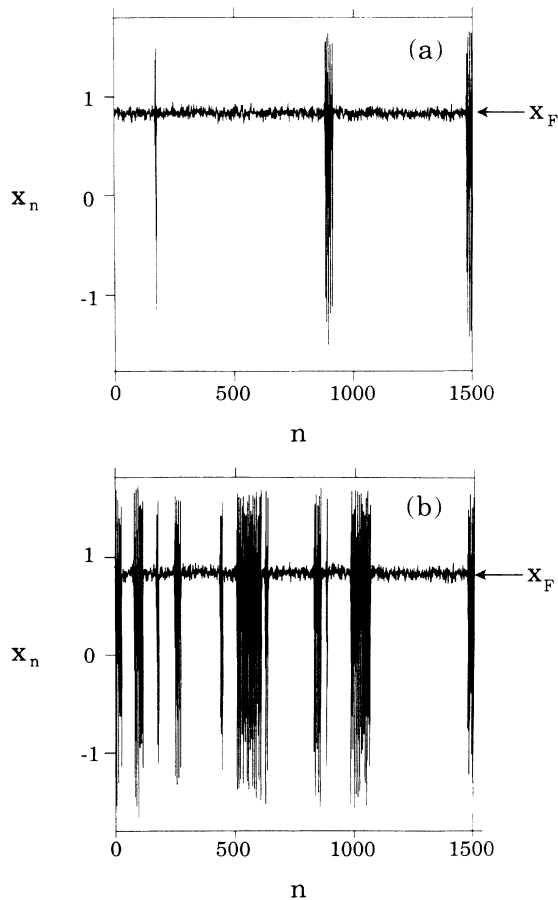


FIG. 2. x_n vs n for two cases with the same realization of the random vector δ . $p_* = 0.2$ and $A_0 = 1.29$ for both cases. (a) $\epsilon = 3.5 \times 10^{-2}$; (b) $\epsilon = 3.8 \times 10^{-2}$.

good although there are significant variations about the general power-law trend. These are to be expected due to the fractal nature of the attractor and have also been seen in numerical calculations of the pointwise dimension for points on chaotic attractors (cf. Grebogi, Ott, and Yorke¹).

Next, we consider the issue of noise. We add terms $\epsilon\delta_{xn}$ and $\epsilon\delta_{yn}$ to the right-hand sides of the Henon map equations. The random quantities δ_{xn} and δ_{yn} are independent of each other, have mean value 0 and mean-squared value 1 ($\langle\delta_x^2\rangle = \langle\delta_y^2\rangle = 1$), and have a Gaussian probability density. Figure 2 shows orbit plots, x_n vs n for 1500 iterates of the noisy map with parameter perturbations given by (2), for two different noise levels and p_* held fixed at $p_* = 0.2$. As predicted the orbit stays

near the fixed point with occasional bursts into the region far from $\xi = 0$, and these bursts are less frequent for small noise levels.

In conclusion, we have shown that there is great inherent flexibility in situations in which the dynamical motion is on a chaotic attractor. In particular, by using only small (carefully chosen) parameter perturbations it is possible to create a large variety of attracting periodic motions and to choose amongst these periodic motions the one most desirable.⁸

This research was supported by the U.S. Department of Energy (Scientific Computing Staff Office of Energy Research). The computation was done at the National Energy Research Supercomputer Center.

(a) Laboratory for Plasma Research.

(b) Departments of Electrical Engineering and of Physics.

(c) Institute for Physical Science and Technology and Department of Mathematics.

¹The periodic orbits are dense in the attractor [i.e., periodic orbits pass through any neighborhood (however small) of any point on the attractor]. For discussions of the relation of ergodic properties of an attractor to its dense set of unstable periodic orbits, see, for example, C. Grebogi, E. Ott, and J. A. Yorke, *Phys. Rev. A* **37**, 1711 (1988); **36**, 3522 (1987); D. Auerbach *et al.*, *Phys. Rev. Lett.* **58**, 2387 (1987); H. Hata *et al.*, *Prog. Theor. Phys.* **78**, 511 (1987); A. Katok, *Publ. Math. IHES* **51**, 137 (1980); R. Bowen, *Trans. Am. Math. Soc.* **154**, 377 (1971).

²E. Ott, C. Grebogi, and J. A. Yorke, in *Chaos: Proceedings of a Soviet-American Conference* (American Institute of Physics, New York, 1990).

³F. Takens, in *Dynamical Systems and Turbulence*, edited by D. Rand and L. S. Young (Springer-Verlag, Berlin, 1981), p. 230.

⁴N. H. Packard *et al.*, *Phys. Rev. Lett.* **45**, 712 (1980).

⁵G. H. Gunaratne, P. S. Linsay, and M. J. Vinson, *Phys. Rev. Lett.* **63**, 1 (1989).

⁶D. P. Lathrop and E. J. Kostelich, "The Characterization of an Experimental Strange Attractor by Periodic Orbits" (to be published).

⁷C. Grebogi, E. Ott, and J. A. Yorke, *Phys. Rev. Lett.* **57**, 1284 (1986); P. Romeiras, C. Grebogi, E. Ott, and J. A. Yorke, *Phys. Rev. A* **36**, 5365 (1987).

⁸The general problem of controlling chaotic systems, while clearly very important, has, so far, received almost no attention. Two exceptions (which are quite different from our approach) are the papers of Hübner (who typically requires large controlling signals) and Fowler [A. Hübner, *Helv. Phys. Acta* **62**, 343 (1989); T. B. Fowler, *IEEE Trans. Autom. Control* **34**, 201 (1989)].