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STRANGE ATTRACTORS THAT ARE NOT CHAOTIC

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It is shown that in certain types of dynamical systems it is possible to have attractors which are strange but not chaotic. Here we use the word *strange* to refer to the geometry or shape of the attracting set, while the word *chaotic* refers to the dynamics of orbits on the attractor (in particular, the exponential divergence of nearby trajectories). We first give examples for which it can be demonstrated that there is a strange nonchaotic attractor. These examples apply to a class of maps which model nonlinear oscillators (continuous time) which are externally driven at two incommensurate frequencies. It is then shown that such attractors are persistent under perturbations which preserve the original system type (i.e., there are two incommensurate external driving frequencies). This suggests that, for systems of the type which we have considered, nonchaotic strange attractors may be expected to occur for a finite interval of parameter values. On the other hand, when small perturbations which do not preserve the system type are numerically introduced the strange nonchaotic attractor is observed to be converted to a periodic or chaotic orbit. Thus we conjecture that, in general, continuous time systems ("flows") which are not externally driven at two incommensurate frequencies should not be expected to have strange nonchaotic attractors except possibly on a set of measure zero in the parameter space.

1. Introduction and definitions

1.1. *Introduction*

In the study of attractors for dynamical systems, it is often observed that the dynamics of typical orbits on an attractor are chaotic in the sense that nearby orbits diverge exponentially from one another with time. Equivalently, one often says that there is "sensitive dependence on initial conditions". In such cases we say that the attractor is a *chaotic attractor*.

In numerical computations, as well as theoretical constructions, it is also often the case that attractors have nonelementary geometrical properties [1] such as noninteger fractal dimension, Cantor set structure, or nowhere differentiability. Ruelle and Takens [2] have called such attractors "strange".

Thus, in this paper, we shall adopt definitions such that *chaotic* refers to the dynamics on the attractor, while *strange* refers to the geometrical structure of the attractor. Precise definitions of chaotic and strange are given in the next subsection.

In many well-known examples it is the case that chaotic attractors are also strange; e.g., the Hénon map exhibits exponential divergence of neighboring trajectories and it has a Cantor set structure. In other cases, however, chaotic attractors are not strange. For example, the logistic map, $x_{n+1} = rx_n(1 - x_n)$, has chaotic attractors for values of r in a set of positive measure, [3] and these attractors are observed to consist of a finite number of disjoint intervals in $0 \le x \le 1$. (See also Li and Yorke [4] who show that attractors are always of this type for piecewise expanding, piecewise differentiable maps of an interval into itself.) Exam-

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ples are well known (e.g. Anosov maps) where chaotic attractors occupy the full surface of a torus and hence are not strange (a computer picture of this type of attractor is given as fig. 1 in Farmer et al. [1]). Also Grebogi, Ott and Yorke [5] investigated invertible maps of a torus obtained as perturbations of quasiperiodic dynamics and also found chaotic attractors which fully occupy a toroidal surface. Other examples also exist (e.g., a solid region with boundary, as shown in fig. 5.14 in Gumowski and Mira [6]).

On the other hand, the answer to the question, "are there strange attractors which are not chaotic", is currently much less clear. It is one of the purposes of the present paper to address this question. We begin this by seeking insight from a specific example. In particular, we examine maps of the form

$$x_{n+1} = f(x_n, \theta_n), \tag{1a}$$

$$\theta_{n+1} = [\theta_n + 2\pi\omega] \mod 2\pi, \tag{1b}$$

where $f(x,\theta) = f(x, \theta + 2\pi)$. For definiteness we set $\omega = (\sqrt{5} - 1)/2$, the golden mean, but we expect that similar observations could be obtained for any irrational number. While θ_n is a scalar in (1b), we shall consider two cases for eq. (1a); case (i) x_n and f are scalars, and case (ii) $x_n = (u_n, v_n)$ and $f = (f_1, f_2)$ are two-dimensional vectors. As pointed out by Sethna and Siggia [7], such maps might result from a nonlinear oscillator driven at two frequencies, and ω will be irrational if the two frequencies are incommensurate. For example, consider a damped pendulum (or Josephson junction [8]) driven at two frequencies ω_1 and ω_2,

$$\frac{d^2\psi}{dt^2} + \nu\frac{d\psi}{dt} + \Omega^2 \sin\psi = A_1 \cos\omega_1 t + A_2 \cos(\omega_2 t + \theta_0). \tag{2}$$

Using a surface of section at times $\omega_1 t_n = 2n\pi$, a map of the form (1) results with $\omega = \omega_2/\omega_1$. Experimental investigation of systems described by eq. (2), or other systems forced at two frequencies, may be a fruitful line of research. As shown here and in refs. 7 and 9, such systems possess distinctive interesting types of nonlinear dynamical behaviors.

In section 2 we demonstrate two particular choices for $f(x,\theta)$ [cf. eq. (1)] for which it can be shown that a strange nonchaotic attractor exists. We then examine the persistence of strange nonchaotic attractors for perturbations of the map (section 3). It is shown that these attractors persist, if the perturbation preserves the system type; that is, if eq. (1a) is perturbed but eq. (1b) is not. On the other hand, we find that small changes in eq. (1b) destroy strange nonchaotic attractors and cause them to be supplanted by periodic or chaotic orbits. These results lead us to believe that, for systems (such as nonlinear oscillators forced at two frequencies) which conform to eqs. (1), strange nonchaotic attractors may be expected to occur over a finite range of parameter space; but that, in more general systems, such attractors, if they occur at all, probably exist only over a set of measure zero in parameter space. (For example, one-dimensional maps with a quadratic maximum have a strange nonchaotic attractor precisely at the point of the accumulation of period doublings [10], where the Hausdorff dimension of the attractor is ~ 0.538....)

1.2. *Definitions*

We define a chaotic attractor as follows:

Definition. A chaotic attractor is one for which typical orbits on the attractor have a positive Lyapunov exponent.

In the above definition we have used the idea of "typical" orbits on the attractor. That is, we assume that, for *almost any* initial condition in the basin of attraction of the attractor, the largest Lyapunov exponents generated by those (typical) initial conditions exist and are identical. Further we use the following definitions of an *attractor* and a *basin of attraction*:

Definition. An attractor is a compact set with a neighborhood such that, for almost every initial condition in this neighborhood, the limit set of the orbit as time tends to $+\infty$ is the attractor.

Definition. The basin of attraction of an attractor is the closure of the set of initial conditions which approach the attractor as time tends to $+\infty$.

We define a strange attractor as follows:

Definition. A strange attractor is an attractor which is not a finite set of points and is not piecewise differentiable. We say that it is piecewise differentiable if it is either a piecewise differentiable curve or surface*, or a volume bounded by a piecewise differentiable closed surface.

Here a curve in a D-dimensional phase space is parametrically representable by $x = X(\eta)$, where x is a D-dimensional phase space vector, η a scalar variable, and $X(\eta)$ is a continuous vector function of η. Similarly for a surface $x = X(\eta)$ with η of dimension between 2 and D-1.

2. Two examples

In this section we consider two examples of eqs. (1) for which strange nonchaotic attractors occur. For the first example, x is a scalar; while, for the second, x is two dimensional.

2.1. *Example* 1

For the case where x is a scalar, there are two Lyapunov exponents for the map, eqs. (1). One of them, corresponding to eq. (1b), is always zero. The other Lyapunov exponent for the map is

$$h = \lim_{m\to\infty}\left\{\frac{1}{m}\sum_{n=1}^{m}\ln|\partial f/\partial x|_{x_n,\theta_n}\right\}. \tag{3}$$

* If the surface has a boundary, then the boundary must be piecewise differentiable (e.g. a two-dimensional square in a three-dimensional space).

We consider here the case where, for definiteness,

$$f(x,\theta) = 2\lambda(\tanh x)\cos\theta, \tag{4}$$

although these considerations can be applied to a class of f choices. For this case, the θ-axis, i.e. $x=0$, is invariant under the map. Whether the θ-axis is an attractor or not is determined by its stability. If $h>0$ for the $x=0$ orbit, then this orbit will be unstable. To see this, we note that two orbits on $x=0$ maintain a constant separation. Thus, if nearby points diverge from each other exponentially, they can only do so by diverging from the θ-axis which is invariant. To calculate h for the $x=0$ orbit, we make use of the ergodicity of θ for irrational ω to convert a trajectory average to a phase space average. From (3), we obtain for $x=0$

$$h = \frac{1}{2\pi}\int_0^{2\pi}\ln|\partial f/\partial x|_{x=0}\,d\theta, \tag{5}$$

which from (4) yields

$$h = \ln|\lambda|. \tag{6}$$

Thus if $|\lambda|>1$, $x=0$ is not an attractor. On the other hand, from (4) and (1), $|x_n|<2|\lambda|$. Hence the orbit is confined to a finite region of space, and there must be an attractor. Due to the ergodicity in θ, the measure on the attractor generated by an orbit is uniform in θ. On the other hand, consider points on the attractor at $\theta=\pi/2$ and $\theta=3\pi/2$. Since the $\cos\theta$ term is zero for these values of θ, the attractor must contain the points ($\theta=\pi/2+2\pi\omega$, $x=0$) and ($\theta=3\pi/2+2\pi\omega$, $x=0$) and must not contain any points in ($\theta=\pi/2+2\pi\omega$, $x\neq 0$) and ($\theta=3\pi/2+2\pi\omega$, $x\neq 0$). Iterating these points forward, we find that, for all positive integers k, the attractor contains the points ($\theta=\pi/2+2\pi k\omega$, $x=0$) and ($\theta=3\pi/2+2\pi k\omega$, $x=0$) but does not contain any points in ($\theta=\pi/2+2\pi k\omega$, $x\neq 0$) and ($\theta=3\pi/2+2\pi k\omega$, $x\neq 0$). Thus for $|\lambda|>1$, $x=0$, $\theta\in[0,2\pi]$ is not the attractor, but there is a dense set of points in the attractor that are also in $x=0$, $\theta\in[0,2\pi]$. To get an idea

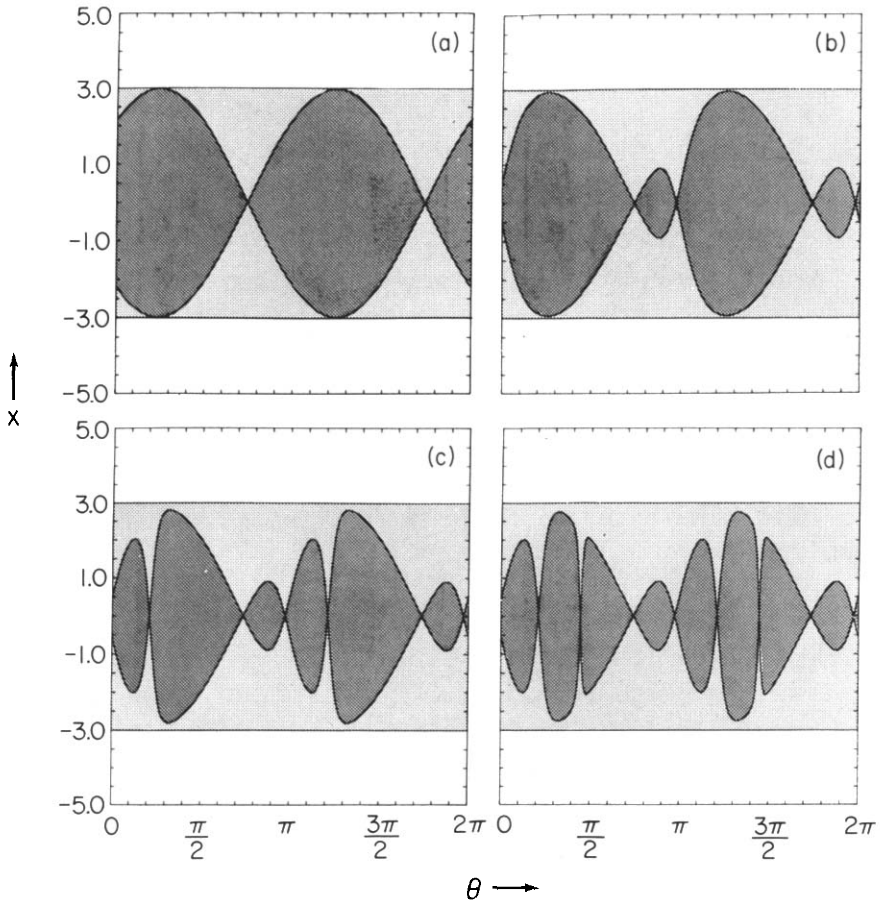


Fig. 1. The strip $|x| < 2\lambda = 3.0$, $\theta \in (0, 2\pi)$ and its (a) first iterate, (b) second iterate, (c) third iterate, and (d) fourth iterate under eqs. (4) and (1).

for the shape of the attractor, consider the strip $|x| < 2|\lambda|$, which we know contains the attractor. Iterates of this strip also contain the attractor. Fig. 1 shows this strip and its first four iterates. As the strip is iterated further, the bounding curves develop more and more zero crossings and these become progressively steeper and steeper. Fig. 2 shows a picture of the attractor for $\lambda = 1.5$ obtained by iterating the map and then plotting the points after the initial transient has died away. From fig. 2 we see that the attractor has points off $x = 0$ (as expected), and, from our previous considerations, it follows that, according to our definition, the attractor is strange. Calculation of the Lyapunov exponent using 3.3×10^4 iterates for the case shown in fig. 2 gives $h \approx -1.059$. Thus the attractor is not chaotic, and we have an example of a strange nonchaotic attractor. In order to prove that h must be negative (implying a nonchaotic attractor), note that $x^{-1} \tanh x \geq \mathrm{d}/\mathrm{d}x(\tanh x)$, with the equality applying only as $x \to 0, \infty$. Thus, from (4), $|\partial f/\partial x| \leq f/x$, or, for x_n and x_{n+1} finite and nonzero,

$$|\partial f/\partial x|_{x_n, \theta_n} < |x_{n+1}/x_n|.$$

Using this in (3) it immediately follows that h is negative, since

$$h < \lim_{m \to \infty} \left\{ \frac{1}{m} \sum_{n=1}^{m} \ln |x_{n+1}/x_n| \right\} = 0, \tag{7}$$

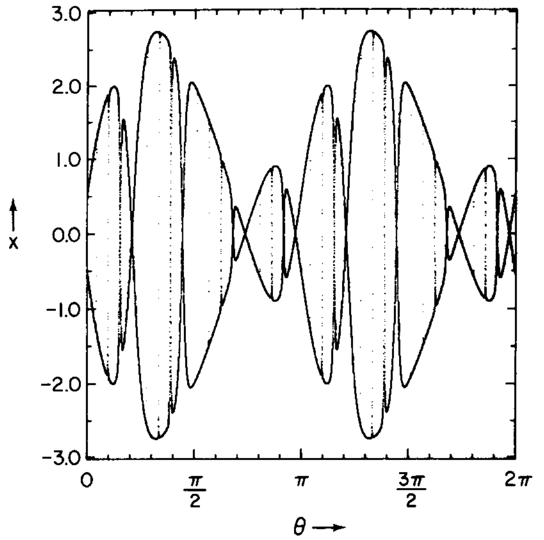


Fig. 2. x versus θ plot of the strange nonchaotic attractor for the two dimensional map given by eqs. (4) and (1) where $\lambda = 1.5$ and the number of iterations is 33,000.

where x_k is assumed to be nonzero. Since the orbit for the strange attractor has $x = 0$ on a set of zero measure (namely, $\theta = \pi \pm \pi/2 + 2\pi k\omega$), the assumption $x_k \neq 0$ in (7) is valid.

2.2. *Example* 2

We now consider a case of a three-dimensional map where eq. (1a) is given by ($x = (u, v)$)

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \frac{\lambda}{1 + u_n^2 + v_n^2} \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}. \quad (8)$$

Note that when $\gamma = 0$, eq. (8) reduces to a form which is similar to (4); viz., $u_{n+1} = \lambda u_n \cos\theta_n / (1 + u_n^2)$. Also note that the θ-axis, i.e. $u = v = 0$, is an invariant curve for the map (8), (1b). As λ is increased from zero $u = v = 0$ will eventually pass from being a stable attractor ($\lambda < \lambda_c$) to being unstable ($\lambda > \lambda_c$). We ask, what happens for $\lambda > \lambda_c$? First we note that there cannot be a continuous attracting invariant curve. In order to see this consider fig. 3. In fig. 3 we illustrate the action of the map (8) and (1b) on the dashed horizontal line shown. It is seen that the image of the line under the map wraps once around the θ-axis in the left-hand sense as shown by the solid line. In fact, for any curve connecting $\theta = 0$ to $\theta = 2\pi$ it is not hard to see that application of the map increases the number of left-hand wraps around $u = v = 0$ by one. Thus there can be no invariant curve except $u = v = 0$. Notice also that if the curve passes through $u = v = 0$ at some θ_0, it would again pass through it at $[\theta_0 + 2\pi n\omega] \bmod 2\pi$ for all integer $n \geq 0$; so the curve would have to be the θ-axis, which contradicts the fact that the θ-axis is unstable.

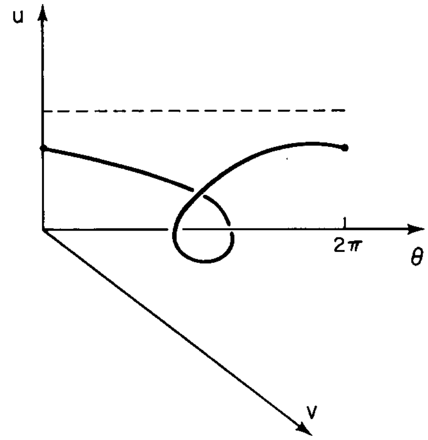
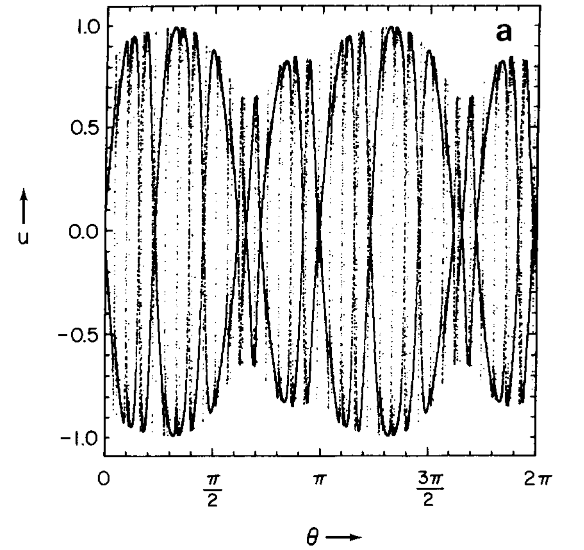


Fig. 3. Effect of applying eqs. (8) and (1b) to a line (schematic). The horizontal dashed line maps to the solid line.

Fig. 4 shows numerical results of iterating eqs. (8) and (1b) for $\gamma = 0.5$ and $\lambda = 2.0$ ($\lambda_c \approx 1.3$ for $\gamma = 0.5$). The calculated Lyapunov exponents are 0 (corresponding to eq. (1b)), -0.124, and -0.380; so the attractor is nonchaotic, yet again it appears to be strange.

To gain further insight into the character of the attractor in fig. 4, we choose a single initial value of θ and pick 100 initial values of (u, v) randomly in $|u| < 1$, $|v| < 1$. We then iterate these 100 points for 1000 steps. Since all initial conditions have the same θ, all the final conditions also have the same θ (cf. eq. (1b)). We find that after 1000 steps the



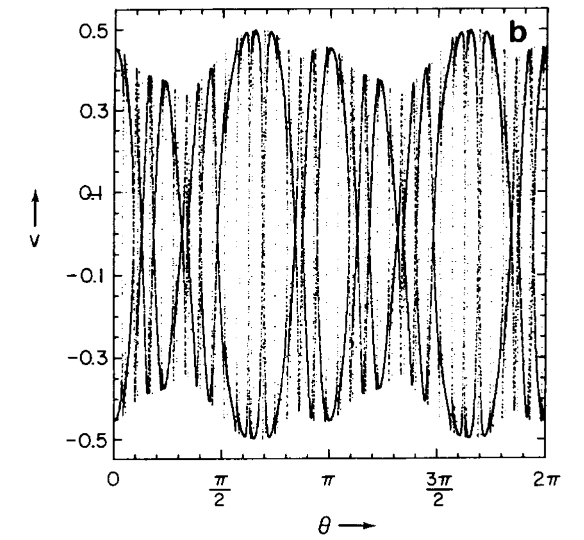


Fig. 4. (a) u versus θ plot of the strange nonchaotic attractor for the three-dimensional map given by eqs. (8) and (1b) where $\gamma = 0.5$ and $\lambda = 2.0$; (b) v versus θ.

(u, v) values for all 100 initial conditions are located at two points symmetric about the θ-axis. Thus, it appears that, for a given value of θ, there are two unique points which attract all initial conditions, $(u, v) = \pm(G_1(\theta), G_2(\theta))$. On the other hand, our previous argument shows that an attractor cannot be a continuous curve. Thus the functions $G_{1,2}(\theta)$ are nowhere continuous, and the attractor is strange.

2.3. *Remarks*

For example 1, if we consider a horizontal curve (as in fig. 3 for example), then successive applications of (4) and (1b) increase the number of times the curve crosses $x = 0$ by two on each iterate. Analogously, for example 2, eqs. (8) and (1b) increase the number of left-hand wraps around $u = v = 0$ by one on each application of the map. Based on these observations, we expect that the length of an initially horizontal curve will, on average, increase linearly with the number of iterates. Thus, we believe that, on average, nearby points diverge from each other linearly with time. This is not inconsistent with our largest Lyapunov number of zero, since the divergence here is slower than exponential.

The above considerations and numerical determinations of the time Fourier transform of the orbits shown in figs. (2) and (4) lead us to offer the speculation that our strange nonchaotic attractors might have a continuum component to the frequency spectrum.

3. The effect of perturbations

In this section we seek some insight into how general the strange nonchaotic attractors of section 2 are. Specifically, we ask two questions:

(i) Within the class of maps of the form (1) (which are applicable to nonlinear oscillators driven at two incommensurate frequencies), do small perturbations of the example maps (section 2) destroy the observed phenomena?

To answer this question, we add perturbations to (4) but not to (1b). We find (section 3.1) that the answer to question (i) above is no. Given that this is so, we ask our second question:

(ii) Do small perturbations which change the system type (by coupling x to eq. (1b)) destroy the phenomena observed in section 2?

As shown in section 3.2, the answer to (ii) is yes.

3.1. *Perturbations that preserve the system type*

The form of $f(x,\theta)$ in eq. (4) has special properties chosen so that a strange nonchaotic attractor could be most easily demonstrated to occur. Here we add perturbations to (4) which destroy the special properties of $f(x,\theta)$ (viz. it is expressible as $f(x,\theta)=f_1(x)f_2(\theta)$ with $f_1(0)=0$, $f_1'(0)=2\lambda$), and then numerically examine the resulting attractor. In particular, we consider

$$f(x,\theta)=2\lambda\tanh x\cos\theta+\alpha_1\cos[\theta+2\pi\beta_1] +\alpha_2 x\cos[\theta+2\pi\beta_2]. \tag{9}$$

We have investigated cases with α_1 and α_2 both simultaneously nonzero, but we shall here only report results for $\alpha_2=0$, since these are representative of what we found in the more general case. We fix $\lambda=1.5$ (as in fig. 2) and $\beta_1=0.125$, and vary α_1. We find that as α_1 is increased from zero the strange nonchaotic attractor of fig. 2 apparently persists, but, for α_1 sufficiently large ($\alpha_1 \gtrsim 0.3$), the attractor becomes nonstrange. Fig. 5a shows a case which appears strange ($\alpha_1=0.2$, $\lambda=1.5$), while fig. 5b, for larger α_1, is a quasiperiodic orbit ($\alpha_1=1.0$, $\lambda=1.5$). To further examine the attractor of fig. 5a, we have considered a large number of initial conditions with the same θ value (as in section 2.2). Upon iteration all these initial conditions converge to the same x-value. Thus, the attractor appears to be in the form $x=G(\theta)$ where $G(\theta)$ is nowhere continuous.

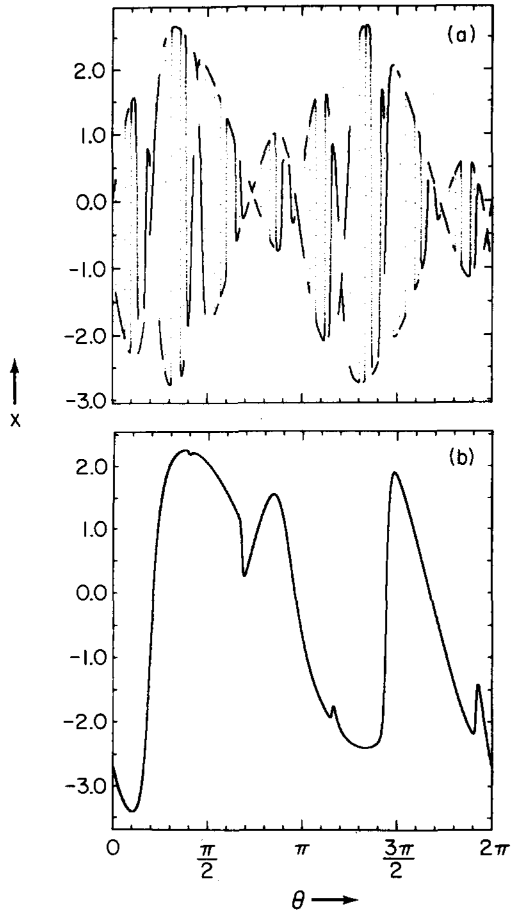


Fig. 5. x versus θ plots illustrating the effect of perturbations that preserve the system type for eqs. (9) and (1) where $\lambda=1.5$, $\alpha_2=0$, and $\beta_1=0.125$. (a) For $\alpha_1=0.2$ the strange nonchaotic attractor appears to persist; (b) for $\alpha_1=1.0$ the strange nonchaotic attractor becomes nonstrange (a quasiperiodic orbit).

3.2. *Perturbations of eq. (1b)*

We now examine the effect of perturbing eq. (1b). We consider

$$\theta_{n+1}=[\theta_n+2\pi\omega+\varepsilon(x+0.5\sin\theta_n)]. \tag{10}$$

We set parameters at those for fig. 5a and examine the effect of finite ε in eq. (10). In all cases tested (about 20 different ε values) a strange nonchaotic attractor did not occur. For example, the smallest $|\varepsilon|$ values tested were $\varepsilon=10^{-4}$ and $\varepsilon=-10^{-4}$. At $\varepsilon=-10^{-4}$ a periodic orbit with a period of about 500 lies close to the original $\varepsilon=0$ strange attractor, while at $\varepsilon=10^{-4}$ a chaotic attractor ($h>0$ in eq. (3)) was observed.

4. Conclusions

The main point of this paper is that strange nonchaotic attractors exist and can occur over a finite range in parameter space for a special type of system. The type of system referred to above is a nonlinear oscillator forced at two incommensurate frequencies (e.g. eq. (2)). Such systems, although in a sense special, are experimentally realizable and are worthy of further study.

Acknowledgements

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