Ergod. Th. & Dynam. Sys. (2003), **23**, 895–906
DOI: 10.1017/S0143385702001360

Characterizing the basins with the most entangled boundaries

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(*Received* 29 *January* 2002 *and accepted in revised form* 5 *September* 2002)

Abstract. In dynamical systems examples are common in which two or more attractors coexist and in such cases the basin boundary is non-empty. The purpose of this paper is to describe the structure and properties of basins and their boundaries for two-dimensional diffeomorphisms. If a two-dimensional basin has a *basin cell* (a trapping region whose boundary consists of pieces of the stable and unstable manifolds of a well-chosen periodic orbit), then the basin consists of a central body (the basin cell) and a finite number of channels attached to it and the basin boundary is fractal. We prove the following surprising property for certain fractal basin boundaries: a basin of attraction B has a basin cell if and only if every diverging path in basin B has the entire basin boundary $\partial \bar{B}$ as its limit set. The latter property reflects a complete entangled basin and its boundary.

1. *Introduction*

One of the goals of dynamical systems is to determine the global structure for ever more complicated dynamical systems. One of these 'global' structures in dynamical systems is the boundaries of basins. For one-dimensional maps, many global structures including basin boundaries are well known; see [**MS**] and references therein. For two-dimensional maps including the Hénon map and the time-2π map of the forced damped pendulum differential equation and forced Duffing differential equation, basins and their boundaries have been studied quite extensively; see, for example [**TK, ML, MGOY, GKOY, Th, AY, KY, S**] and references therein. Experiments and computations indicate that mixing in chaotic flows generates certain coherent spatial structures; see, for example

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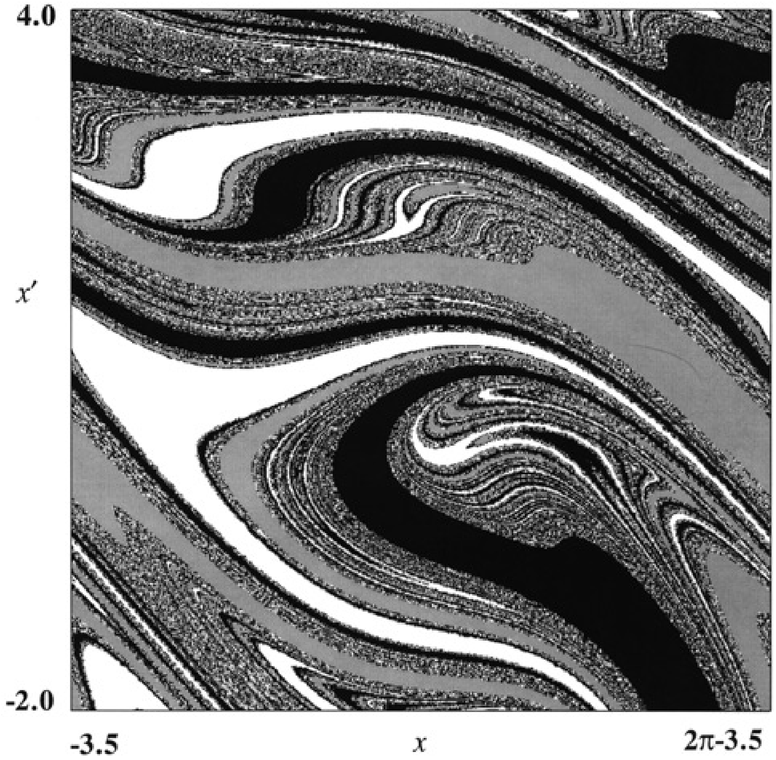
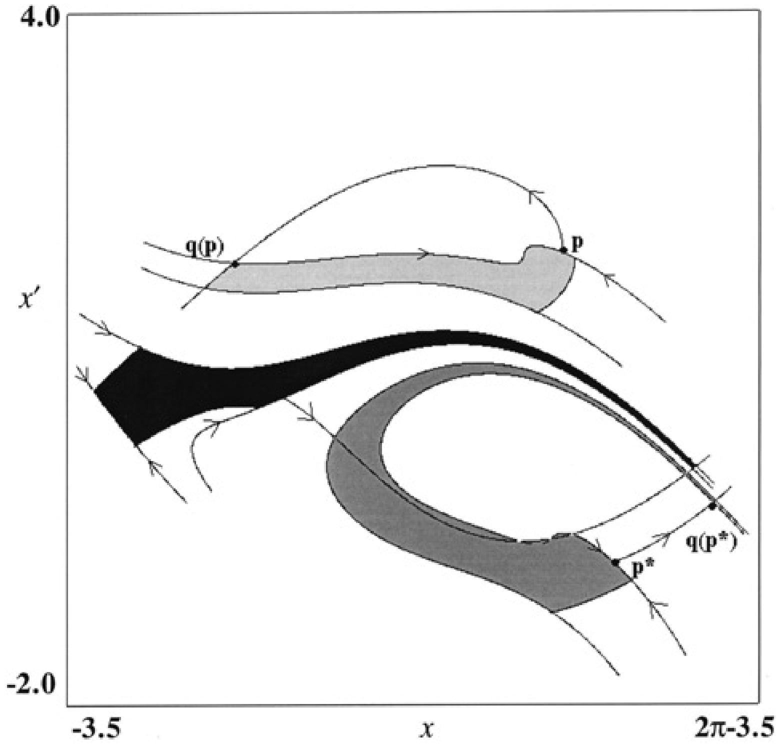


(a)

FIGURE 1. (a) displays three basins of the time–2π map of the forced damped pendulum differential equation $x''(t) + 0.2x'(t) + \sin x(t) = 1.66\cos(t)$. They are grey, white and black. All three basins have basin cells which are shown in (b). The uppermost basin cell is in the grey region, the middle basin cell is in the white basin; the lower basin cell is in the black basin. The middle basin cell is generated by a period-3 orbit and so it has six sides and is diffeomorphic to Figure 2. The two other basins in (a) have a basin cell generated by a saddle-hyperbolic period-2 orbit and so have four sides. A basin cell determines both the structure of its basin and the global structure of the corresponding basin boundary. In the uppermost basin cell, one of the two period-2 points has been labeled by p, and its corresponding primary homoclinic point $q(p)$ is a corner point of the basin cell. Similarly, in the lower basin cell, one of the two period-2 points has been labeled by p^*, and its corresponding primary homoclinic point $q(p^*)$ is a corner point of the basin cell. The result in this paper implies that for each of the basins the limit set of any diverging path in that basin is the basin's entire boundary.

[**SO, GAMCA**]. Sometimes, these structures are related to the manifolds of hyperbolic periodic points [**GAMCA**]. The basin cells introduced in [**NY1, NY2**] allow us to discuss the global structure of basin boundaries for many choices of parameters in well-known two-dimensional maps including the Hénon map and the time-2π map of the forced damped pendulum differential equation.

Let M denote either $\mathbb{R}^2$ or the cylinder $\mathbb{R} \times S^1$. Let $F : M \to M$ be a C^1-diffeomorphism. In Figure 1(a), parts of three basins of attraction are shown. Each basin is unbounded with infinite area. The basins shown are open, connected, simply connected sets. These basins are highly convoluted. We can characterize such a convoluted set B by examining paths Γ in B, that is, Γ a continuous map from [0, 1) into B. When B is



(b)

FIGURE 1. Continued.

an unbounded set, we say that a path Γ in B is *diverging* if $d_B(\Gamma(0), \Gamma(t)) \to \infty$ as $t \to 1$, where $d_B(\cdot, \cdot)$ is a path metric in B; see §2 for its definition. If B is any of the three basins of Figure 1(a) (which are unbounded sets), there exist diverging paths in B. Our main result in this paper implies that each basin B shown in Figure 1(a) has a surprising property: each diverging path in B 'limits on $\partial\bar{B}$' (that is, for every $p \in \partial\bar{B}$ and every neighborhood U of p, $\Gamma([0, 1))$ intersects U). We want to point out that there is an open set of C^1-diffeomorphisms $g : M \to M$ which have unbounded basins with this property. Our goal is to determine when basins B have the property that each diverging path in B 'limits on $\partial\bar{B}$'.

1.1. *Basin of a trapping region.* A basin (for F) is usually defined to be the set of points x for which $\omega(x)$ is contained in a specified compact attractor. Of course, the attractor is contained in a compact trapping region. In this paper we take a slightly more general approach of emphasizing the role of a trapping region. By a *compact region* we mean a simply connected, connected compact set with a non-empty interior. We say that a compact region Q is a *trapping region* (for F) if $F(Q) \subset Q$ and $F(Q) \neq Q$. (Note that we do not require that $F(Q)$ is in the interior of Q.) If Q is a trapping region, then we

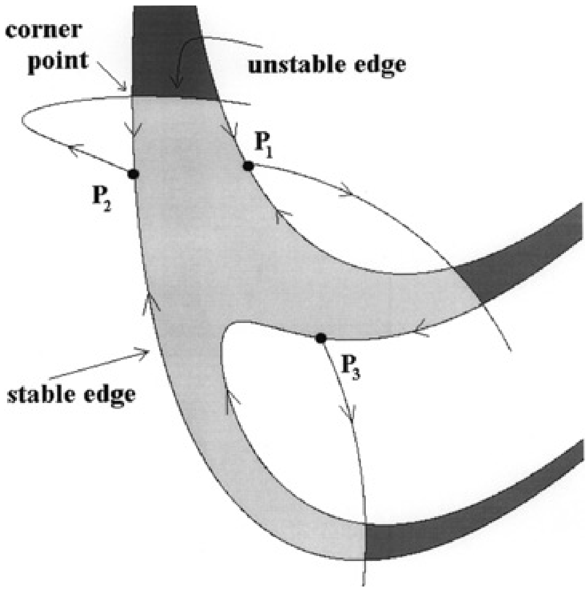


FIGURE 2. A *basin cell* (as shown in light grey) for a two-dimensional diffeomorphism F is a trapping region whose boundary consists of pieces of the stable and unstable manifolds of some period-m orbit P. In this figure, $m = 3$. It follows that this basin cell has $2m$ sides, namely m *stable edges* and m *unstable edges* (pieces of the stable and unstable manifolds of the points of P). We also say that the orbit P *generates* the basin cell. Each of the three dark grey regions is in the basin and is the initial part of a channel (see text) of the basin. The basin has a fractal basin boundary since three of the six corner points of the basin cell are homoclinic points.

define the *basin of* Q to be the set of points which eventually map into the interior of Q. In this paper, a set B is a *basin* if it is the basin of some trapping region. This modified definition avoids the problem of determining the attractors of a system. Note that the union of finitely many basins is a basin. For our choices of a trapping region Q for the black basin in Figure 1(a), there is an attracting period-2 orbit in Q and also a saddle fixed point. If a basin of attraction contains an attracting fixed point and no other attracting periodic orbits, then the trapping region may include periodic saddles or even invariant Cantor sets, so the orbits of some points in the basin will not converge to the fixed-point attractor. Let B denote a basin. A point $x \in M$ is a *boundary point* of B if $x \in \bar{B} \setminus \mathrm{Int}(\bar{B})$. The *boundary* of B is the set $\partial \bar{B} = \bar{B} \setminus \mathrm{Int}(\bar{B})$. (Note that this concept of boundary is slightly different from the notion of the topological boundary, ∂B.) We say that $\partial \bar{B}$ is a *fractal basin boundary* if it contains a transversal homoclinic point. Note, in particular, that the closure of the homoclinic points of a periodic orbit (such as the orbit P in the theorem) is a Cantor set.

1.2. *Basin cell and main result.* When we introduced the notion of 'basin cell' [**NY1, NY2**] (see also Figure 2 and its caption), we showed that when such cells exist, fractal basin boundaries can be characterized robustly. Robust structures (that is, those structures that persist under small (smooth) perturbations in the system) are particularly valuable in studying nonlinear dynamics where many structures (like chaotic attractors) can often be destroyed by arbitrarily small (smooth) perturbations. In our approach to the theory of basins, we examine a trapping region whose boundary consists of alternating pieces of the stable and unstable manifolds of certain saddle periodic points. These periodic orbits are said to *generate* this trapping region. If a single periodic orbit generates a trapping region, then it is called a *basin cell*; see §3 for some properties. A typical trapping region which is a basin cell is shown in Figure 2. Only a few (if any) of the infinitely many periodic orbits in the boundary of the basin may generate a basin cell. Of course, these periodic orbits are boundary points of the basin cell. We say that a basin B *has a basin cell* if B is the basin of a trapping region that is a basin cell. Basin cells reveal a great deal about the structure of the corresponding basin. For example, the six-sided basin cell of Figure 2 (the light grey region) is generated by a periodic orbit of period 3. The corresponding basin can be viewed as the central body (basin cell) plus three channels (dark grey) that connect to it. These channels are infinitely long and wind in a very complicated pattern without crossing each other. The channels may vary greatly in thickness but must occasionally get quite thin as they wander back and forth (see the three regions in Figure 1(a)). Our theorem states that under mild hypotheses, if B is a basin, then B has the property that the limit set of each diverging path $\Gamma : [0, 1) \to B$ is the boundary $\partial \bar{B}$ if and only if B has a basin cell (see §2 for the theorem).

Remarks. Figure 1(a) displays three basins, B_1 (white region), B_2 (grey region) and B_3 (black region). B_1 has a basin cell (the middle basin cell shown in Figure 1(b)) that is diffeomorphic to the light grey object in Figure 2. Each of the two other basins B_2 and B_3 has a basin cell generated by a period-2 orbit. These are the uppermost basin cell and lower basin cell in Figure 1(b). However, there are many basins that have no basin cell. For example, if the basin's boundary is not fractal, then it follows that there are no homoclinic points on the boundary and there is no basin cell. Previously, we exploited basin cells as a tool for proving whether a basin B is a Wada basin. (Basin B is a Wada basin if every basin boundary point of B is also the boundary point of at least two other basins.) In [**NY2**] we showed by utilizing properties of basin cells that each of the three basins B_1, B_2 and B_3 in Figure 1 is a Wada basin and that the boundaries of all three basins coincide. Hence, the three basins have the property that each neighborhood of each point on the boundary of any of the basins intersects all three basins. Therefore, applying the result of this paper, the limit set of every unbounded path in any of the three basins B_1, B_2 or B_3 equals $\partial \bar{B}_1 = \partial \bar{B}_2 = \partial \bar{B}_3$.

Overview. The organization of the paper is as follows. The main result for characterizing fractal basin boundaries is stated in §2. Section 3 contains preliminaries and reviews some properties of basin cells. The proof of the theorem is given in §4. All the pictures in this paper were made using *Dynamics* [**NY3**].

2. *Statement of main theorem*

Let all of the tangent spaces of M be equipped with an Euclidean inner product $\langle \cdot , \cdot \rangle$. Let $F : M \to M$ be a C^1-diffeomorphism. Let $B \subset M$ be an open, simply connected, connected set. Let $\Gamma : [0, 1) \to B$ be a half-open path. We say that $q \in M$ is a *limit point of* Γ, if for every open neighborhood U of q and every $0 < \varepsilon < 1$, there exists t such that $1 - \varepsilon \leq t < 1$ and $\Gamma(t) \in U$. We call the collection of all limit points of Γ, the *limit* *set of* Γ. A point $x \in M$ is *B-accessible* if and only if $x \in \partial \bar{B}$ and there exists a path γ: $[0, 1) \to \operatorname{Int} \bar{B}$ (the interior of $\bar{B}$) such that $\lim_{t\to 1} \gamma(t) = x$, that is, the limit set of γ is a single point.

In the introduction, we stated that the path Γ in B is diverging if $d_B(\Gamma(0), \Gamma(t)) \to \infty$ as $t \to 1$. We now specify what we mean by this. A definition of a diverging path requires the notion of distance between two points in B. For any differentiable path $\phi : [0, 1] \to B$, define the length $\ell(\phi)$ of ϕ by

$$\ell(\phi) = \int_0^1 |\phi'(s)|\, ds = \int_0^1 \sqrt{\langle \phi'(s), \phi'(s) \rangle}\, ds.$$

For every pair of points $p, q \in B$, define the 'path distance' $d_B(p, q)$ between p and q to be the infimum of $\ell(\phi)$ taken over all C^1-paths ϕ in B having $\phi(0) = p$ and $\phi(1) = q$. Two points p, q in B might be close in the usual sense but every path lying entirely in B might be quite long, so $d_B(p, q)$ would be large. The distance $d_B(\cdot, \cdot)$ defined on B is a (path) metric. By the path Γ in B being *diverging* we mean that $\lim_{t\to 1} d_B(\Gamma(0), \Gamma(t)) = \infty$, that is, for every $K > 0$ there exists $0 < t_K < 1$ such that for all $t \in (t_K, 1)$, $d_B(\Gamma(0), \Gamma(t)) > K$. Note that this definition does not imply that $d_B(\Gamma(0), \Gamma(t_1)) < d_B(\Gamma(0), \Gamma(t_2))$ for all $0 < t_1 < t_2 < 1$. We now state our main result, which was reported (without proof) in [**NY4**].

THEOREM. *Let $F : M \to M$ be a C^1-diffeomorphism for which there exists a connected, simply connected, trapping region T. Let $B \subset M$ be the basin of the trapping region T such that:*

(a) *$\bar{B} \neq T$ and $\bar{B} \neq M$;*
(b) *there exists exactly one B-accessible periodic orbit P;*
(c) *the orbit P is saddle-hyperbolic; and*
(d) *the orbit P has a homoclinic point, and the stable manifold of P has no tangency with the unstable manifold of P.*

Then B has a basin cell if and only if (1) *there are diverging paths in B, and* (2) *for every diverging path* Γ *in B, the limit set of* Γ *equals the boundary of B.*

3. *Preliminaries*

Let $F : M \to M$ be a C^1-diffeomorphism. Let p be a saddle-hyperbolic fixed point of F, that is, the eigenvalues λ and μ of the Jacobian matrix $DF(p)$ satisfy $|\lambda| < 1 < |\mu|$. The *stable manifold* $W^s(p)$ of p is the set $W^s(p) = \{x \in M : F^n(x) \to p \text{ as } n \to \infty\}$, and the *unstable manifold* $W^u(p)$ of p is the set $W^u(p) = \{x \in M : F^n(x) \to p \text{ as } n \to -\infty\}$. A point $q \in M$ is a *homoclinic point* with respect to p if and only if (a) $q \neq p$, and (b) $\lim_{n\to\infty} F^n(q) = p$ and $\lim_{n\to\infty} F^{-n}(q) = p$. For $x, y \in W^s(p)$, we denote the closed segment in $W^s(p)$ with end points x and y by $S_p[x, y]$. For $x, y \in W^u(p)$, we denote the closed segment in $W^u(p)$ with end points x and y by $U_p[x, y]$. A homoclinic point q of F with respect to the fixed point p is called a *primary homoclinic point* if and only if $S_p[p, q] \cap U_p[p, q] = \{p, q\}$. Primary homoclinic points always exist whenever there exist homoclinic points, see Palis and Takens [**PT**]. In our proof of the theorem, we use the Lambda Lemma due to Palis (1969); see [**PT**].

LAMBDA LEMMA. *Let F be a C^1 diffeomorphism of M with a saddle-hyperbolic fixed point p, and let D be an arc in $W^s(p)$. Let A be an arc meeting $W^u(p)$ transversally at some point q. Then $\bigcup_{n\geq 0} F^{-n}(A)$ contains arcs arbitrarily C^1-close to D.*

Remark. If there is a transversal homoclinic point $q \in W^u(p) \cap W^s(p)$, then A can be chosen to be an arc in $W^s(p)$. In this case the Lambda Lemma implies that $W^s(p)$ accumulates on itself.

3.1. *Basin cells.* For clarity, we repeat the brief definition in the introduction, adding important details. For any basin, there are many choices for a trapping region, but there is, at most, one periodic orbit that generates a basin cell for that basin. If a periodic orbit generates a basin cell, there are a countable number of ways of choosing its basin cell. A *cell* is a connected, simply connected, compact region such as a disk. A cell C is called a *manifold cell* if the boundary of C is piecewise smooth and there exists a saddle-hyperbolic periodic orbit P such that (a) the boundary of C consists alternately of pieces of the stable manifold $W^s(P)$ and unstable manifold $W^u(P)$ of the periodic orbit P, and (b) every point $x \in \partial C$ that is on both the stable and unstable manifolds of P is a point of transverse intersection of $W^s(P)$ and $W^u(P)$. See Figure 2 for $m = 3$. In this case, we also say that the cell C is a *manifold cell for P*, the cell C is *generated by the orbit P* or, also, the orbit P *generates* the manifold cell C.

For a manifold cell C, each of the sides of C that is in the stable manifold of P is called a *stable edge* of C and each of the sides of C that is in the unstable manifold of P is called an *unstable edge* of the cell C. The common point of a stable and an unstable edge of a cell C is called a *corner point* of the cell C. Note that each of the corner points of a cell C generated by a period-m orbit P is (1) a periodic point, (2) a primary homoclinic point, or (3) a homoclinic point which is a heteroclinic point for F^m ($m \geq 2$). Note that for a fixed point p, if p is not a corner point of a cell C then both corner points are homoclinic points, one on each stable manifold branch.

In this paper, a *basin cell* is a manifold cell C which is a trapping region, so $F(C) \subset C$ and $F(C) \neq C$. Hence, the basin consists of one component. We call a cell C a *single branch basin cell* if and only if (a) C is a basin cell and (b) there exists an unstable edge U_0 of C such that the unstable edges of C are U_0 and its iterates $U_k = F^k(U_0)$, $1 \leq k \leq m-1$. The majority of basin cells we encountered are single-branch basin cells.

In the proof of the theorem, we use the following results on basin cells. The first result gives a criterion that guarantees that the manifold cell generated by a saddle-hyperbolic periodic orbit is a basin cell. The second result states that the accessible boundary points of the basin of a basin cell are all on the stable manifold of the periodic orbit in the boundary of the basin that generates the cell. We start by defining some notions related to paths in a basin.

Let B be a basin for the map F. We refer to a continuous map Γ from $[0, 1)$ into B as a path Γ in B. In addition to paths in the basin B, we consider continuous maps $\gamma : [0, 1] \to M$ such that $\gamma(x) \subset \text{Int}\, \bar{B}$ for $0 < x < 1$, $\gamma(0) \in \partial \bar{B}$ and $\partial(1) \in \partial \bar{B}$. We say that γ is a *cross-cut* of basin B if and only if $\gamma(0) \neq \gamma(1)$ and γ is one-to-one.

BASIC BASIN CELL PROPOSITION. *Let $P = \{p_k\}_{1 \le k \le m}$ be a saddle-hyperbolic periodic orbit of F that generates a manifold cell C. Assume that C satisfies the following conditions:*

(a) *P is contained in ∂C;*

(b) *for every integer k $(1 \le k \le m)$, p_k is not a corner point of C;*

(c) *F maps each of the unstable edges of C into C; and*

(d) *C has $2m$ edges (that is, m stable and m unstable edges).*

Then C is a basin cell.

Proof. All that is needed is to show that C is a trapping region. For a detailed proof, see [**NY2**]. □

ACCESSIBILITY PROPOSITION. *Let P be a saddle-hyperbolic periodic orbit of F which generates a manifold cell C. Assume that C is a basin cell. Let B be the basin of the cell C. If $P \subset \partial \bar{B}$, then every B-accessible point is on the stable manifold $W^s(P)$ of P.*

Proof. For a proof, see [**NY2**]. □

4. *Proof of the theorem*

Let the map F, the basin B and the periodic orbit P of period m $(m \in \mathbb{N})$ satisfy (a)–(d) of the theorem. A region R in B is called *bounded* if it contains no image of a diverging path (that is, there exists no diverging path $\Gamma : [0, 1) \to B$ such that $\mathrm{Im}(\Gamma) = \{\Gamma(t) : 0 \le t < 1\} \subset R$), and it is *unbounded* if it is not bounded. Let B^{acc} denote the union of the interior of $\bar{B}$ and the B-accessible points. The set B^{acc} is the completion of the interior of $\bar{B}$ in the path metric. We note that if B is the basin of the basin cell generated by P, then the interior of $\bar{B}$ coincides with B, so B^{acc} is the union of the basin B and the B-accessible points and is the completion of B in the path metric.

To prove the '$\Longrightarrow$' implication of the theorem, we need some lemmas.

ASSUMPTION. *In the 'Stable Branch Proposition' and the 'Existence of Diverging Paths Lemma' that follow we assume that B has a basin cell C.*

STABLE BRANCH PROPOSITION. *Let $p \in P$, and let $W^s_+(p)$ and $W^s_-(p)$ denote the two branches of $W^s(p) \setminus \{p\}$. Then the closure of each of the stable manifold branches equals $\partial \bar{B}$, that is, $\overline{W^s_+(p)} = \overline{W^s_-(p)} = \partial \bar{B}$.*

Proof. For any $p' \in P$, let $W^s_+(p')$ and $W^s_-(p')$ denote the two branches of $W^s(p') \setminus \{p'\}$. We write $\overline{W^s_+(p')}$ and $\overline{W^s_-(p')}$ for the closures of $W^s_+(p')$ and $W^s_-(p')$, and $\overline{W^s(p')}$ for the closure of the stable manifold of p'. Let $p \in P$ be given.

Let U be an unstable edge of the basin cell C. Let $p_0 \in P$ be such that $U \subset W^u(p_0)$. (Note that if $m = 1$, then $p_0 = p$.) Write $p_n = F^n(p_0)$ with $0 \le n \le m - 1$. We write $F(\overline{W^s_+(p_n)}) = \overline{W^s_+(p_{n+1})}$ and $F(\overline{W^s_-(p_n)}) = \overline{W^s_-(p_{n+1})}$, where $0 \le n \le m - 1$ and $p_m = p_0$. Note that $\overline{W^s(p_n)} = \overline{W^s_+(p_n)} \cup \overline{W^s_-(p_n)}$ $(0 \le n \le m - 1)$. Assume that the end points of U are in $W^s_+(p_i)$ and $W^s_-(p_j)$ for some $0 \le i, j \le m - 1$.

First assume that $m = 1$. Note that $p_0 = p_i = p_j$. Applying the Lambda Lemma gives $\overline{W^s(p_0)} \subset \overline{W^s_+(p_i)} = \overline{W^s_+(p_0)}$ and $\overline{W^s(p_0)} \subset \overline{W^s_-(p_j)} = \overline{W^s_-(p_0)}$. Hence, $\overline{W^s_+(p_0)} = \overline{W^s_-(p_0)} = \overline{W^s(p_0)}$.

From now on, we assume that $m \geq 2$. Note that $U \subset W^u(p_0)$ connects $\overline{W^s_+(p_i)}$ and $\overline{W^s_-(p_j)}$ with $i \neq j$. Applying the Lambda Lemma we have that $\overline{W^s(p_0)} \subset \overline{W^s_+(p_i)}$ and $\overline{W^s(p_0)} \subset \overline{W^s_-(p_j)}$. Apply F^i to both sides of U and applying the Lambda Lemma gives $\overline{W^s(p_i)} \subset \overline{W^s_+(p_{2i})}$. Repeatedly applying F^i to both sides of the edge U gives $\overline{W^s(p_0)} \subset \overline{W^s_+(p_i)} \subset \overline{W^s(p_i)} \subset \overline{W^s_+(p_{2i})} \subset \overline{W^s(p_{2i})} \subset \cdots$. This sequence of sets is periodic and, in particular, each of the elements is equal to $\overline{W^s(p_0)}$ so $\overline{W^s_+(p_{ni})} = \overline{W^s(p_{ni})} = \overline{W^s(p_0)}$ for $n = 0, 1, 2, \ldots$. Similarly, repeatedly applying F^j to both sides of the unstable edge U gives $\overline{W^s_-(p_{nj})} = \overline{W^s(p_{nj})} = \overline{W^s(p_0)}$ for $n = 0, 1, 2, \ldots$. Hence, $\overline{W^s_+(p_i)} = \overline{W^s(p_i)} = \overline{W^s(p_j)} = \overline{W^s_-(p_j)}$.

Let $S(p_n)$ be the stable edge of C containing p_n with $0 \leq n \leq m-1$. For every r, t $(0 \leq r, t \leq m-1, r \neq t)$ such that the stable edges $S(p_r)$ and $S(p_t)$ are connected by an unstable edge of C, we have, by the previous argument, that either $\overline{W^s_+(p_r)} = \overline{W^s(p_r)} = \overline{W^s(p_t)} = \overline{W^s_-(p_t)}$ or $\overline{W^s_+(p_t)} = \overline{W^s(p_t)} = \overline{W^s(p_r)} = \overline{W^s_-(p_r)}$. Since ∂C is connected, we can proceed around the edges of the cell and so $\overline{W^s_+(p_n)} = \overline{W^s_-(p_n)} = \overline{W^s(p_n)} = \overline{W^s(p_0)}$ for all n, $0 \leq n \leq m-1$. Therefore, all these sets are equal. Hence, $\overline{W^s_+(p)} = \overline{W^s_-(p)} = \overline{W^s(p)}$. Since $P \subset \partial\bar{B}$, by the Accessibility Proposition, every B-accessible point is on the stable manifold of the orbit P which generates C. Since the accessible points are dense in the boundary, we get that $\overline{W^s_+(p)} = \overline{W^s_-(p)} = \partial\bar{B}$. □

EXISTENCE OF DIVERGING PATHS LEMMA. *There exist diverging paths* $\Gamma : [0, 1) \to B$.

Proof. Let U_0 be an unstable edge of C running between the stable manifold branches $W^s_+(p_a)$ and $W^s_-(p_b)$ of P. Let $D_0 \subset B^{\rm acc}$ be the component of $B^{\rm acc} \setminus U_0$ which contains no points of P. Note that the ends of U_0, denoted a_0 and b_0, are in $\partial\bar{B}$, so U_0 is a cross-cut of B. By assumption, a_0 and b_0 are not periodic points. For each $n \in \mathbb{N}$, let U_n and D_n be defined by $U_n = F^{-nm}(U_0)$ and $D_n = F^{-nm}(D_0)$, and write $a_n = F^{-nm}(a_0)$ and $b_n = F^{-nm}(b_0)$ for the end points of U_n. Since U_0 is a cross-cut of B, and a_n and b_n are in two stable manifold branches of the periodic orbit P, U_n is a cross-cut of B for all n.

Let $Q_1 = D_0 \setminus D_1$, and for $n \in \mathbb{N}$ we define the quadrilateral Q_n by $Q_n = D_{n-1} \setminus D_n$. For $n \in \mathbb{N}$, let $\Gamma_n : [0, 1] \to Q_n$ be a path such that (a) $\Gamma_n(0) \in U_{n-1}$; (b) $\Gamma_n(1) \in U_n$; and (c) $\Gamma_n(0) = \Gamma_{n-1}(1)$. The two stable manifold branches $W^s_+(p_a)$ and $W^s_-(p_b)$ have a non-empty intersection with Q_n for all n.

Intersection Property. For every $x \in W^s_-(p)$ and every $\varepsilon > 0$, $B(x; \varepsilon)$ intersects $\mathrm{Im}(\Gamma_n)$ for all but finitely many n. The proof of this property, which is like the proof of the Lambda Lemma, is left to the reader.

Let $\Gamma : [0, 1) \to B$ be defined by $\Gamma(t) = \Gamma_n(n^2t + nt + 1 - n^2)$ for $1 - (1/n) \leq t < 1 - 1/(n+1)$, where $n \in \mathbb{N}$. Let $x_a, x_b \in W^s_-(p)$ such that $x_a \neq x_b$. Then it follows from the Intersection Property that $d_B(\Gamma(0), \Gamma(t)) \to \infty$ as $t \to 1$. Therefore, Γ is a diverging path. □

Proof of '$\Longrightarrow$' in the theorem. We have proved that there is a diverging path and that its limit set is $\partial\bar{B}$. Essentially that proof works for any diverging path (with minor changes that are left to the reader). Hence, the limit set of each diverging path $\Gamma : [0, 1) \to B$ equals $\partial\bar{B}$. This completes the proof. □

To prove the '$\Longleftarrow$' implication of the theorem, we need some lemmas.

ASSUMPTION. *In the Cross-cut Existence Lemma, Unbounded Component Lemma and Cross-cut Basin Cell Lemma, we assume that* (1) *there are diverging paths in B, and* (2) *for every diverging path* $\Gamma : [0, 1) \to B$*, the limit set of* Γ *equals the boundary of B.*

CROSS-CUT EXISTENCE LEMMA. Let $p \in P$. There exists an arc CC in $W^u(p) \cap B^{\mathrm{acc}}$ such that CC is a cross-cut of B that connects two different stable manifold branches of the orbit P.

Proof. Let $p \in P$ be given. By assumption (d) of the theorem, the periodic point p has a homoclinic point. Hence, p has a primary homoclinic point [**PT**]. Let $q \in W^s(p)$ be a primary homoclinic point with respect to p such that $W^u(p)$ and $W^s(p)$ cross each other at q. Let $S[p, q]$ be the arc in $W^s(p)$ having p and q as end points and let $U[p, q]$ be the arc in $W^u(p)$ connecting p and q. The union of $S[p, q]$ and $U[p, q]$ is a closed curve. Let R be the region bounded by $S[p, q]$ and $U[p, q]$. Note that R may intersect the basin but is not contained in B^{acc}. Let $r \in S[p, q] \setminus \{p, q\}$ be a point on the stable branch of p. Let s be any point that is on the stable manifold of the B-accessible periodic orbit P and is outside $\bar{R}$. Since the stable and unstable manifolds of the B-accessible periodic P have no tangencies, both r and s are B-accessible. Let $\gamma : [0, 1) \to B$ be any diverging path. By assumption, such a Γ exists and the limit set of Γ equals $\partial \bar{B}$. Hence, for every $\varepsilon > 0$, $\mathrm{Im}(\Gamma)$ intersects $B(r; \varepsilon)$ and $B(s; \varepsilon)$, where $B(x; \varepsilon)$ is the open ball (centered at x) with radius ε and so $\mathrm{Im}(\Gamma)$ must cross $U[p, q]$. Thus, $\mathrm{Im}(\Gamma)$ crosses the unstable manifold $W^u(p)$ of p.

Let $D_0 \subset B^{\mathrm{acc}}$ be an unbounded component of $B^{\mathrm{acc}} \setminus K$ (for some cross-cut K of B connecting two stable manifold branches) such that $\mathrm{Im}(\Gamma) \subset D_0$. It is no restriction to assume that D_0 contains no points of P, and there are two stable manifold branches of P that have a non-empty intersection with D_0. Let S_a and S_b be the two stable branches that 'straddle' $\mathrm{Im}(\Gamma)$, that is, the end points of each cross-cut of B that crosses $\mathrm{Im}(\Gamma)$ and connects two stable branches are contained in S_a and S_b.

Let $0 < \varepsilon_1 < ||p-q||/100$ be given. Let $t_1 > 0$ be such that $\Gamma(t_1) \in U[p, q] \cap B(p; \varepsilon_1)$. Define $\varepsilon_2 = ||p - \Gamma(t_1)||/2$, and let $t_2 > t_1$ be such that $\Gamma(t_2) \in U[p, q] \cap B(p; \varepsilon_2)$. Define $\varepsilon_3 = ||p - \Gamma(t_2)||/2$, and let $t_3 > t_2$ be such that $\Gamma(t_3) \in U[p, q] \cap B(p; \varepsilon)$. Since the stable manifold branches S_a and S_b straddle $\mathrm{Im}(\Gamma)$, there is a compact arc CC in $U[p, \Gamma(t_1)] \setminus U[p, \Gamma(t_3)] \cap B^{\mathrm{acc}}$ such that CC is a cross-cut of B and CC connects the stable manifold branches S_a and S_b. This implies that there exists an arc CC in $W^u(p) \cap B^{\mathrm{acc}}$ such that CC is a cross-cut of B and CC connects two different stable manifold branches. □

UNBOUNDED COMPONENT LEMMA. *Let* $p \in P$*. Let* $CC \subset W^u(p)$ *be a cross-cut of B that connects two different stable manifold branches. Then* $B^{\mathrm{acc}} \setminus CC$ *has an unbounded component* D_0 *that contains no point of P.*

Proof. Let the periodic point p and cross-cut CC be as in the lemma. Then $B \setminus CC$ consists of two components, denoted D_0 and D_1, where $p \in D_1$. Let $J = \bigcap_{n=0}^{\infty} F^n(T)$, where T is the trapping region of B. The sequence $\{F^n(T)\}_{n=0}^{\infty}$ is a nested sequence of non-empty, compact and connected sets and the boundary of $F^n(T)$ is piecewise smooth for all n. Hence, J is non-empty, compact and connected. We now want to show that J does not intersect CC. For every $n \in \mathbb{N}$, $F^n(CC)$ does not intersect CC, since $CC \subset W^u(p)$ and the endpoints of CC are in two different stable manifold branches. This implies that J does not intersect CC. Hence, J is non-empty, compact, connected and does not intersect CC.

Let p_i $(1 \leq i \leq m)$ be the points of P. Note that there exist no other B-accessible periodic points. Let U_i be the unstable inward branch of B-accessible periodic point p_i. Then U_i has the following properties: (i) U_i contains points in B; (ii) U_i has limit points in J; and (iii) U_i does not cross CC. Hence, all U_i and p_i are in the same component of $B^{\mathrm{acc}} \setminus CC$ as J. Since $p \in D_1$ so are J, all U_i, and all p_i. Hence, D_0 contains no B-accessible periodic points.

We now show that D_0 is unbounded. Suppose that D_0 is bounded. Then $F^{-i}(D_0)$ is bounded for $1 \leq i \leq m$. Since $F^{-m}(D_0) \subset D_0$ and P is the only B-accessible periodic orbit, we have that $B^{\mathrm{acc}} \setminus \bigcup_{i=0}^{m-1} F^{-i}(D_0)$ is also bounded. This latter set contains all points of P. It follows that there exist no diverging path $\Gamma : [0, 1) \to B$. This contradicts the assumption that there exist diverging paths $\Gamma : [0, 1) \to B$. □

CROSS-CUT BASIN CELL LEMMA. *Let the cross-cut CC and the unbounded component D_0 be as in the Unbounded Component Lemma. Then $B^{\mathrm{acc}} \setminus \bigcup_{n=0}^{m-1} F^n(D_0)$ is a basin cell.*

Proof. Let the cross-cut CC and the unbounded component D_0 be as in the Cross-cut Basin Cell Lemma. Write $CC_0 = CC$ and for $0 \leq k \leq m-1$, define $CC_k = F^k(CC_0)$. Note that CC_k is a cross-cut of B^{acc}, where $0 \leq k \leq m-1$. Hence, for every k $(0 \leq k \leq m-1)$, there is an arc S_k in $W^s(F^k(p))$ such that the union of the arcs $\bigcup_{k=0}^{m-1}(S_k \cup CC_k)$ in the stable and unstable manifolds of P constitute a closed curve. Since B is simply connected, the bounded region enclosed by this closed curve is a manifold cell C.

Let U be an unstable edge of C. Thus $U = CC_k$, for some $0 \leq k \leq m-1$. Since $U \subset B^{\mathrm{acc}}$, $F(U) \subset B^{\mathrm{acc}}$. By construction, either $F(U)$ is an unstable edge of C or $F(U)$ does not intersect any of the unstable edges of C. Let a and b be the end points of U. Obviously, $F(a) \in C$, $F(b) \in C$. If $F(a)$ is a corner point of C, then $F(b)$ is a corner point of C and $F(U)$ is an unstable edge of C. If $F(a)$ is not a corner point of C then $F(b)$ is not a corner point of C and $F(U) \subset C$, so $F(U)$ is not an unstable edge of C. Since U is an arbitrary unstable edge of C, we now have that every unstable edge of C is mapped into C under F (and also $F(\partial C) \subset C$). Since the stable and unstable manifolds of P have no tangencies, by the Basic Basin Cell Proposition, C is a basin cell. In fact, C is a single branch basin cell. □

This completes the proof of the theorem. □

Acknowledgements. This work was in part supported by the National Science Foundation (Division of Mathematical Sciences and Physics), and by the W. M. Keck Foundation.

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