Math 452 Homework 10

An Overview of Material Covered
Counts as a double homework assignment

1 (

5 points) **True or False:** Every continuous map in the plane from the unit circle to itself has at least one fixed point.

2 (

10 points) Let f be a continuous map from [0,1] to [0,1]. **True or False:** Every non-attracting fixed point displays sensitive dependence on initial conditions.

3 (

10 points) Consider the map $f(x) = (x + \frac{1}{3}) mod 1$ on [0,1). Notice that 0 has period 3 but the map has no chaotic orbits. **Explain** why this does not validate *Period 3 Implies Chaos.*

4 (

15 points) Let f be a continuous map from [0,1] **onto** [0,1]. (That is $f([0,1]) = [0,1]$.) Prove the following:

a. f must have at least one fixed point

b. f^2 must have at least two fixed points. (Hint: Explain why either f or f^2 must have points $0 <= x_1 < x_2 <= 1$ such that $f(x_1) = 0$ and $f(x_2) = 1$.)

c. If in addition 0 and 1 are not fixed points for f, show that f^2 must have at least 3 fixed points.

5 (

15 points) Consider the following map:

$$f(x) = Ax$$

where

$$A = \begin{pmatrix} -2 & 2\frac{1}{2} \\ -5 & 5\frac{1}{2} \end{pmatrix}$$

Please answer the following:

a. Define what is meant by the stable and unstable manifold of (0,0).

b. Determine which points (x,y) belong to the stable and unstable manifold of the origin.

c. Compute the Lyapunov Numbers of the origin. (Hint: Find eigenvalues and eigenvectors.)

d. Describe the stability of the origin and what happens to the orbits of neighboring points.

6 (

15 points) Let $A = [0, \frac{1}{3}], B = [\frac{1}{3}, \frac{2}{3}], C = [\frac{2}{3}, 1]$

$$\text{Let } f(x) = \begin{cases} x + \frac{1}{3}, & \text{if } x \in A \bigcup B \\ 3 - 3x, & \text{if } x \in C \end{cases}$$

a. Describe the trajectories starting at the endpoints of A, B and C.

b. Show the transition graph.

c. List all the distinct itineraries of periodic orbits for periods up to and including 5. (Be careful not to list different itineraries that have the same orbit.) List the ititneraries in alphabetical order. When there is more than one representation of the same orbit, list only the one that comes earlier in alphabetical ordering.

d. Create a periodic table for the map (as in Table 1.3).

e. Compute the Lyapunov Exponents of an orbit of period 5. Be sure to describe which transition intervals correspond to the orbit.

7 (

15 points) Consider the function $f(x)$ defined on $[0, 1] \times [0, 1] \times [0, 1]$ in R^3

$$f(x) = \begin{cases} (\frac{1}{2}x, \frac{1}{2}y, (2z) mod 1) & \text{if } 0 \leq z \leq \frac{1}{2} \\ (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y + \frac{1}{2}, (2z) mod 1) & \text{if } \frac{1}{2} < z \leq 1 \end{cases}$$

a. Describe the resulting attractor

b. Compute the Lyapunov dimension of the attractor

c. Comute the box dimension of the attractor

8 (

15 points) Compute the box counting dimension of the following sets:

a. $S = \{0\} \bigcup \left\{ \frac{1}{k^p} : k \geq 1 \right\}$ for $p \geq 1$ fixed.

b. $S = \{0\} \bigcup \left\{ \frac{1}{p^k} : k \geq 1 \right\}$ for $p > 1$ fixed.

where k is an integer (i.e, $k = 1, 2, 3, ...$). **Hint:** To help in solving the general cases above, consider the two special cases below

- $S = \{0\} \bigcup \left\{ \frac{1}{2^k} : k \geq 1 \right\}$
- $S = \{0\} \bigcup \left\{ \frac{1}{k^2} : k \geq 1 \right\}$