2. EXAMPLES

2.1a PLOTTING DIRECTION FIELD AND TRAJECTORIES

The purpose of this example is to plot some trajectories together with the
direction field of the Lotka/Volterra differential equations (IV)
\[ x' = (c_1 + c_3 x + c_5 y)x, \quad y' = (c_2 + c_4 y + c_6 x)y \]
where the parameter values are given by \( c_1 = 0.5, \ c_2 = 0.5, \ c_3 = -0.0005, \ c_4 = -0.0005, \ c_5 = -0.00025, \) and \( c_6 = -0.00025. \)

\[ \text{dynamics-win} \leftarrow \text{Start the program Dynamics-Win;} \]
\[ \text{de} \leftarrow \text{fetch the differential equations menu;} \]
\[ \text{lv} \leftarrow \text{get the Lotka/Volterra equations;} \]
\[ \text{pm} \leftarrow \text{fetch the parameter menu and verify the values above;} \]
\[ \text{dem} \leftarrow \text{fetch the differential equations menu;} \]
\[ \text{df} \leftarrow \text{and plot the direction field.} \]
A resulting picture is given in Figure 2-1a.

For plotting trajectories, mark the initial conditions by a small cross.
\[ \text{c} \leftarrow \text{clear the screen;} \]
\[ \text{km} \leftarrow \text{fetch the kruis (cross) menu,} \]
\[ \text{kkx} \leftarrow 5 \text{ 5} \leftarrow \text{and set the scale kks of the permanent cross to be 5 pixels by 5 pixels;} \]

To get continuous trajectories, connect the consecutive computed dots.
\[ \text{wwm} \leftarrow \text{fetch the when and what to plot menu, and} \]
\[ \text{con} \leftarrow \text{connect consecutive dots;} \]
\[ \text{p} \leftarrow \text{pause the program;} \]
\[ \text{t} \leftarrow \text{plot the trajectory.} \]
\[ \text{i} \leftarrow \text{kk} \leftarrow \text{initialize and draw a permanent cross at y;} \]
\[ \text{<space bar>} \leftarrow \text{hit the <space bar> to un-pause;} \]
\[ \text{<e,>,>,>,>}, \leftarrow \text{use the arrow keys to move the small cross to a new initial condition for a trajectory, and} \]
\[ \text{p} \leftarrow \text{pause the program.} \]
Repeat the 4 previous steps (i \leftarrow 1 \text{ kk} \leftarrow, \text{<space bar>}, <e,>,>,>,>,, p \leftarrow). A resulting picture is given in Fig. 2-1b.

Note. Instead of going through the 4 steps repeatedly, an easy way to initialize is to move the mouse arrow to the desired position and then\ double click the left mouse button. To have a cross drawn at this position of y1, enter command kk1. You can plot several trajectories simultaneously (command tn, or commands tnb and tn), but no crosses can be plotted.
Figure 2-1a: Direction field

This figure shows a direction field for the Lotka/Volterra differential equations

\[ x' = (c_1 + c_3x + c_5y)x \]
\[ y' = (c_2 + c_4y + c_6x)y \]

where \( c_1 = 0.5 \), \( c_2 = 0.5 \), \( c_3 = -0.0005 \), \( c_4 = -0.0005 \), \( c_5 = -0.00025 \), and \( c_6 = -0.00025 \).

Topic of discussion. This figure suggests there are "at least" four equilibria. What are they?
Figure 2-1b: Trajectories of an ODE

This figure shows some trajectories for the Lotka/Volterra differential equations, with parameters as in Figure 2-1a. The initial value of each trajectory is indicated with a small cross.

Topic of discussion. There is an asymptotically stable equilibrium where the trajectories converge. What does this picture suggest about the eigenvalues and eigenvectors of that equilibrium?
Figure 2-1c: Direction field

This figure shows a direction field for the Lotka/Volterra differential equations

\[ x' = (c_1 + c_3x + c_5y)x \]
\[ y' = (c_2 + c_4y + c_6x)y \]

where \( c_1 = 4, c_2 = 6, c_3 = -1, c_4 = -1, c_5 = -1, \) and \( c_6 = -3. \)

If you wish, you may change the grid (command \texttt{fg}, which is in the Actions, Hints and Options menu for \texttt{df}). The default setting of a (15 by 15) grid (\( \texttt{fg} = 15 \)) has been used in creating this picture.

Topic of discussion. This figure suggests there are "at least" four equilibria. What are they?
Figure 2-1d: Trajectories of an ODE

This figure shows some trajectories for the Lotka/Volterra differential equations, with parameters as in Figure 2-1c. \( T_n = 10 \) trajectories are plotted simultaneously from \((10,0)-(10,10)\), \((0,10)-(10,10)\) and \((1.0)-(1.0)\).

Topic of discussion. There are two asymptotically stable equilibria where the trajectories converge. What does this picture suggest about the eigenvalues and eigenvectors of those equilibria?

To create a part of this picture, carry out the steps in Example 2-1b through entering the step size. Then enter the following commands.

\[
\begin{align*}
\text{tn} & \leftarrow 10 \quad \text{plot 10 trajectories simultaneously if t is entered;} \\
\text{diag} & \leftarrow \quad \text{set ya and yb to the opposite corners of the screen,} \\
\text{yy} & \leftarrow \quad \text{that is, ya = (0,0) and yb = (10,10);} \\
\text{sv} & \leftarrow \text{sva} \leftarrow \quad \text{get the y vectors and verify the settings for ya, yb;} \\
\text{sva} & \leftarrow 10 \quad \text{set vector and set vector ya;} \\
\text{t} & \leftarrow \quad \text{select sva0 to set coordinate \#0 of ya equal to 10;} \\
& \quad \text{plot the 10 trajectories starting at ya-yb.}
\end{align*}
\]
2.1b PLOTTING DIRECTION FIELD AND TRAJECTORIES

The purpose of this example is to plot some trajectories together with the
direction field of the Lotka/Volterra differential equations (iv)

\[ x' = (c1 + c3*x + c5*y)*x, \quad y' = (c2 + c4*y + c6*x)*y \]

where the parameter values are given by \( c1 = 4, \quad c2 = 6, \quad c3 = -1, \quad c4 = -1, \quad c5 = -1, \) and \( c6 = -3, \) and the scales of the variables \( 0 \leq x, \quad y \leq 10. \) To get
reliable numerical results, set the step size of the DE-solver to 0.0001.

\[
\text{dynamics-win } \leftarrow \quad \text{Start the program } \text{Dynamics-Win};
\]
\[
de \leftarrow \quad \text{fetch the differential equations menu};
\]
\[
\text{lrv } \leftarrow \quad \text{get the } \text{Lotka/Volterra} \text{ equations};
\]
\[
\text{pm } \leftarrow \quad \text{fetch the parameter menu};
\]
\[
c1 \leftarrow 4 \quad \text{set the value of } c1 \text{ to } 4;
\]
\[
c2 \leftarrow 6 \quad \text{set the value of } c2 \text{ to } 6;
\]
\[
c3 \leftarrow -1 \quad \text{set the value of } c3 \text{ to } -1;
\]
\[
c4 \leftarrow -1 \quad \text{set the value of } c4 \text{ to } -1;
\]
\[
c5 \leftarrow -1 \quad \text{set the value of } c5 \text{ to } -1;
\]
\[
c6 \leftarrow -3 \quad \text{set the value of } c6 \text{ to } -3;
\]
\[
xS \leftarrow 0 \quad 10 \quad \text{set the } x \text{ scale to run from } 0 \text{ to } 10;
\]
\[
yS \leftarrow 0 \quad 10 \quad \text{set the } y \text{ scale to run from } 0 \text{ to } 10;
\]
\[
dem \leftarrow \quad \text{fetch the differential equations menu};
\]
\[
\text{step } \leftarrow 0.0001 \quad \text{set the } \text{step size } \text{to} \quad 0.0001;
\]
\[
df \leftarrow \quad \text{and plot the direction field.}
\]

A resulting picture is given in Figure 2-1c.

For plotting trajectories, mark the initial conditions by a small cross.
\[
c \leftarrow \quad \text{clear the screen};
\]
\[
\text{km } \leftarrow \quad \text{fetch the kruis (cross) menu,}
\]
\[
\text{kks } \leftarrow 5 \quad 5 \quad \text{and set the scale kks of the permanent cross to be } 5 \text{ pixels by } 5 \text{ pixels.}
\]

To get continuous trajectories, connect the consecutive computed dots.
\[
\text{wwm } \leftarrow 1 \quad \text{fetch the when and what to plot menu, and}
\]
\[
\text{con } \leftarrow \quad \text{connect consecutive dots.}
\]
\[
p \leftarrow \quad \text{pause the program;}
\]
\[
t \leftarrow \quad \text{plot the trajectory.}
\]

Either carry out repeatedly the 4 steps (i \( \leftarrow \) \( \text{kk} \), \( <\text{space bar}>\), \( <\leftarrow, \downarrow, \rightarrow, \uparrow>\), \( p \leftarrow 1 \)) in Example 2-1a or use the mouse (move the mouse arrow
to the desired position and then double click the left mouse button to initialize and enter command \( \text{kk1} \) to draw a cross at this position of \( y1 \).)
You can plot several trajectories simultaneously (commands \text{tab} and \text{tn}), but
no crosses can be plotted. A resulting picture is given in Fig. 2-1d.
2.2 BASINS OF ATTRACTION

The basin of an attractor is the set of points whose iterates tend to that attractor. In this example we want to plot all the basins and attractors for the Henon map (h)

\[(x,y) \rightarrow (r\text{ho} - x^2 + c^1 \cdot y, x)\]

where the parameter values are given by \(r\text{ho} = 0.5\) and \(c^1 = 0.9\).

dynamics-win \(\leftarrow\) start the program Dynamics-Win;
\(h \leftarrow\) select the Henon map and the main menu appears;
\(p\text{m} \leftarrow\) select the parameter menu;
\(r\text{ho} \leftarrow 0.5 \leftarrow\) set the value of \(r\text{ho}\) to 0.5;
\(c^1 \leftarrow 0.9 \leftarrow\) set the value of \(c^1\) to 0.9;
\(<\text{Esc}>\) retrieve the parameter menu and then the main menu;
\(n\text{em} \leftarrow\) select the numerical explorations menu;
\(b\text{m} \leftarrow\) fetch the basin of attraction menu;
\(b\alpha \leftarrow\) plot the basins and attractors.

A 100 by 100 grid of points will be tested. The colors used to color the grid boxes indicate something that is in the box. Initially all grid boxes are uncolored but as the routine \(b\alpha\) proceeds, all grid boxes will eventually be colored. The \(b\alpha\) routine selects an uncolored grid box and tests the box by examining the trajectory that starts at the center of the grid box. The trajectory may or may not pass through grid boxes that have previously been colored, but what it encounters will determine how the initial grid box is colored. First, pixels are colored dark blue, indicating that the trajectories of the centers of these boxes diverge. After a while, some boxes are colored light blue followed by boxes which are colored green. The dark blue region is the basin of infinity. There is one other attractor, namely an asymptotically stable period 2 orbit (green). The basin of the period 2 orbit is colored light blue.

If you print the current picture on paper, you just get a solid black plot. The program has the feature to erase a color. The basin of the period 2 attractor is colored with color number 3.

\(c\text{m} \leftarrow\) fetch the color menu;
\(e\text{m} \leftarrow\) select the erase color menu;
\(e^3 \leftarrow\) erase color 3.

The resulting picture is similar to Figure 2-2.

*Note.* The resolution of the picture that appears on the screen, is low. By first invoking a command like \(r\)h (high resolution), the command \(b\alpha\) generates a picture of high resolution.

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Figure 2-2: Basins of attraction and Attractors

The black area and white areas are two basins of the Henon map (h)

\[(x,y) \rightarrow (0.5 - x^2 + 0.9y, x)\]

The black area shows points whose trajectories diverge (to infinity). There are two attractors for this choice of parameters. The routine `ba` plots all attractors and their basins. It has plotted the basin of infinity, and the basin of a stable period 2 orbit. The white area is the basin of attraction of the period 2 attractor, and is obtained by erasing the color of the basin of the period 2 attractor. We used command `e3` to erase color 3. The isolated dots (in the white region) are on the period 2 attractor. Its basin has been erased and so is white.

There are several ways for finding out what kind of simple attractors may exist. The easiest way may be the following. Plot the basin in high resolution (commands `rh`, `ba`). After the picture has been completed, enter command `cnt` to count the pixels and observe there are two pixels colored with color 2 indicating that a period-2 attractor exists.
2.3 PLOTTING TRAJECTORY VERSUS TIME

The purpose of this example is to plot the trajectory (iterates 1 through 100) of \( x_0 = 0.4 \) versus "time" of the logistic map (log)

\[
x \rightarrow 3.83 \times x(1-x)
\]

dynamics-win \rightarrow \quad \text{start the program Dynamics-Win;}
log \rightarrow \quad \text{select the logistic map and the main menu appears;}
\rho \rightarrow 3.83 \quad \text{set the value of } \rho \text{ to 3.83;}
\text{con} \quad \text{turn the toggle con on to connect consecutive dots;}

Set Vector y1 to be 0.4:

\sv \rightarrow \quad \text{set vector;}
\sv1 \rightarrow \quad \text{set vector y1;}
\sv10 \rightarrow 0.4 \quad \text{select sv10 to set coordinate #0 of y1 equal to 0.4;}

Notice that if you are using the menu, you should now respond with ok twice by hitting <Enter> twice.

The routine t (trajectory) will plot the time (the number of the iterate) on the horizontal axis if the "plot time" toggle pt has to be turned on. See the when and what to plot menu for the current status of pt.

\pm \rightarrow \quad \text{fetch the parameter menu, and then}
\ww \rightarrow \quad \text{get the when and what to plot menu;}
\pt \quad \text{turn the plot time toggle on.}
\text{<Esc> \rightarrow \quad get the parameter menu;}
\xs \rightarrow 0 100 \quad \text{set the x scale to run from 0 to 100;}
i \rightarrow \quad \text{initialize;}
t \quad \text{plot the trajectory}
xax1 \rightarrow \quad \text{draw the x axis with tic marks;}

The resulting picture is similar to Figure 2-3.
Figure 2-3: Trajectory versus time

This figure shows a trajectory versus time of the logistic map (log)

\[ x \rightarrow 3.83 \times x (1-x) \]

The initial condition of the trajectory is 0.4.

Topic of discussion. Using only this graph and the map, identify where there is a fixed point. What can you conclude about the map near the fixed point?
2.4 BIFURCATION DIAGRAM: plotting trajectory versus parameter

The purpose of this example is to plot a trajectory (iterates 61 through 260) versus a parameter of the logistic map (log)

\[ x \rightarrow \rho x * x(1-x) \]

where \( \rho \) varies from 2 to 4. For the minimum value of \( \rho \) (\( \rho = 2 \)) and some initial condition \( x_0 \), first calculate the first 60 iterates of \( x_0 \) without plotting (that is, calculate \( x_k \) with \( 1 \leq k \leq 60 \) without plotting); we say the number of pre-iterates is 60. In Dynamics this number is called bifurcation pre-iterates (bifpi is listed in bifm and its default value is 60). Then plot the next 200 iterates of \( x_0 \) (that is, plot \( x_k \) with \( 61 \leq k \leq 260 \)). In Dynamics this number is called bifurcation dots (bifd is listed in bifm and its default value is 200). Increase \( \rho \) slightly, say by 0.01 to \( \rho_{\text{new}} \). If the trajectory is not reinitialized at \( \rho_{\text{new}} \), we say the attractor is followed and take for the initial condition at the current value \( \rho_{\text{new}} \) the last point that was plotted using the previous parameter value \( \rho_{\text{prev}} \). Then calculate 60 iterates of this point without plotting and then plot the next 200 iterates. Increase \( \rho \) again, calculate 60 iterates of the last computed point at the previous parameter value without plotting and then plot the next 200 iterates. Continue increasing the parameter \( \rho \) until it assumes the maximum value of 4.

Note. In Dynamics, when creating a bifurcation diagram, the attractor is followed if bif is set to 0 (default setting) and the bifurcation parameter is increased by the value of bifr divided by the value of bifv.

\[
\text{dynamics-win} \leftarrow \text{start the program} \; \text{Dynamics-Win};
\text{log} \leftarrow \text{select the logistic map and the main menu appears;}
\text{nem} \leftarrow \text{fetch the numerical explorations menu, and then}
\text{bifm} \leftarrow \text{get the bifurcation diagram menu, and verify that}
\pi = 60 \; \text{and} \; \text{bifd} = 200.
\]

The command bifr in the bifurcation diagram menu is for specifying the range of the bifurcation parameter \( \rho \). The default setting is from not set to not set.

\[
\text{bifr} \leftarrow 2 \; 4 \leftarrow \text{make the parameter} \; \rho \; \text{vary from} \; 2 \; \text{to} \; 4;
\leftarrow \text{retrieve the bifurcation diagram menu;}
\text{bifs} \leftarrow \text{plot the bifurcation diagram on the screen.}
\]

The resulting picture is called a bifurcation diagram and is similar to Figure 2-4.
Figure 2-4: Bifurcation diagram created on the screen

In this bifurcation diagram for the logistic map (log)

\[ x \rightarrow \rho x(1-x) \]

the parameter \( \rho \) varies from 2.0 to 4.0, that is, the bifurcation range (bifr) is from 2.0 to 4.0. This bifurcation diagram is created on the screen using command bifs.

There are two different ways of plotting a bifurcation diagram. If the bifurcation diagram is sent directly to the printer (command bif), then you can set the length of the diagram and you have an option of printing the parameter values (bifp is on). If you create the bifurcation diagram on the screen, then you can set the size of the picture (commands high and wide in scm), but parameter values cannot be printed.

Frequently, in bifurcation diagrams the parameter is varied along the horizontal axis, while one of the space coordinates is on the vertical axis. You can obtain such a graphical representation by rotating the picture three times 90 degrees clockwise (command rot).

30  Dynamics