A Chaos Lemma

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1. INTRODUCTION. Many physical systems that are easy to describe exhibit complicated behavior. A reasonable goal is to describe the different kinds of behavior a system can have.

In a study attempting to determine if the solar system is stable, Poincaré described the first “chaotic” system, known as the “restricted three body problem”. Here three bodies move in a plane under gravitational forces as described by Newton. One of the bodies is of negligible mass, and does not influence the other two. Those two move in circles about the center of mass. Poincaré’s ideas were applied by NASA to send a spacecraft with minimal fuel through the tail of a comet. In this application the three bodies were the Earth, the Moon, and a spacecraft. The spacecraft had been launched for other purposes and was parked about 1 million miles from the Earth in the direction of the Sun, collecting information. Near the end of the spacecraft’s mission, the Jacobi-Zinner comet was discovered to be heading toward the Sun in a trajectory that would take it near the earth’s orbit (passing about 60 million miles from the earth). Even though the spacecraft had enough fuel remaining for only small orbit corrections, Dr. Robert Farquhar realized after extended computer simulations that a tiny push could put the spacecraft on a complicated trajectory that would intercept the comet’s tail; see Figure 1. Poincaré had made it clear that in such chaotic systems, there were infinitely many trajectories, each with long term behavior that was quite different from the others.

Figure 1. THE FARQUHAR TRAJECTORY. The initial change in velocity was on 6/10/82. The other dates, all in 1983, denote near passes of the Moon.

Farquhar’s trajectory was started with a firing of the rockets that changed the velocity of the craft by only 10 miles per hour. On this trajectory, the spacecraft had five near passes of the Moon, finally passing about 60 miles from the surface of the Moon. This last pass was sufficient to kick the spacecraft out of the Earth-Moon neighborhood and toward the rendezvous with the comet. The spacecraft made the first direct measurements of the composition of a comet’s tail.
We denote Poincaré’s two large masses by $E$ and $M$. The third body of negligible mass is denoted by $S$ (for spacecraft). Our approach is slightly different from Poincaré’s. First, with the origin as the center of mass, we represent the position of $S$ relative to $E$ and $M$ using polar coordinates $r, \theta$, with $\theta = 0$ denoting the direction of $M$ and $r = 1$ the distance of $M$ from the origin. In these “rotating” coordinates, $E$ and $M$ remain fixed; see Figure 2.

![Figure 2. A trajectory is shown intersecting set B with the distance from O decreasing.](image)

Mathematicians and scientists often use objects called “symbol sequences” to describe the different kinds of trajectories a chaotic system can have. For an example of this in the $E$-$M$-$S$ system, we let $S_A$ denote the set of points with $r = 1$ and $0 < \theta < 0.1$, and let $S_B$ denote the set of points with $r = 1$ and $2\pi - 0.1 < \theta < 2\pi$. If $p_n$ denotes the $n$th element of the associated symbol sequence, for the Earth-Moon-Spacecraft system (which is not quite our ideal system) $p_n$ is in $S_A$ if the Moon’s $n$th inward crossing is ahead of the Moon; it is in $S_B$ if the crossing is behind the Moon. The Farquhar trajectory would be described by the symbol sequence $ABABB$, after which the craft left the Earth-Moon environment. Most trajectories ($p_n$) that don’t collide with the Earth or the Moon would leave the Earth-Moon system after a finite number of symbols. Nonetheless some trajectories would yield an infinitely long sequence ($p_n$) of the symbols $A$ and $B$, and there are uncountably many such sequences that do not repeat in any periodic pattern.

We present a simple method of studying chaos that considers all the possible sequences that different trajectories can have when passing through some collection of sets. In our example of sets, $S_A$ and $S_B$, representing parts of the Moon’s orbit, we have two symbols, but other situations require more sets.

The spacecraft problem presents an opportunity to explain why discrete time maps can be interesting. We investigate maps, not differential equations. Poincaré argued that the motion could be best understood as a map on a two-dimensional space. Each time the body $S$ (with position written as $(\theta(t), r(t))$), crosses the circle $r = 1$ with derivative $r' \leq 0$, we record two numbers: $\theta(t)$ and its derivative $\theta'(t)$. The trajectory of $S$ has a generalized total energy that is constant, which we assume is known. Knowing this energy and $\theta$, $\theta'$, and having $r = 1$, we can solve for the derivative $r'$. Hence, we can solve the differential equation until the next crossing if there is one. If $p_n := (\theta, \theta')_n \in S^1 \times \mathbb{R}$ denotes the $n$th such crossing, we can write

$$p_{n+1} = f(p_n).$$
To evaluate $f$, we must solve the differential equation numerically. Using this kind of map, Poincaré showed that the behavior of some trajectories is very complicated for some ratio of the masses of $M$ and $E$.

We do not analyze the E-M-S system further, but rather explore a framework that is applicable to a wide variety of problems. Chaos is a fairly well understood phenomenon in dynamics: consequences of its presence often include the “curses” of unpredictability and practical incomputability of the long-term behavior in a system governed by deterministic dynamics [1]. This property is usually formulated in terms of “sensitive dependence on initial conditions”, which we define in Section 4.

Next we analyze the celebrated Smale Horseshoe [9].

2. THE SMALE HORSESHOE. SMALE HORSESHOE CONSTRUCTION. To construct a simplified Smale Horseshoe, begin with a unit square $Q := ABCD$ in the plane, as pictured in Figure 3. Write $Q = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Next let $S_1$ and $S_2$ denote two disjoint horizontal strips in $Q$, and let $Q_0$ denote the union $S_1 \cup S_2$ of the two strips as shown in Figure 3. The horseshoe map $f$ sends each horizontal strip affinely onto a vertical strip. The map can be thought of as a map from the plane to itself, or a map from $Q$ into the plane, or a map from $Q_0$ into $Q$; here we care only about those points in $Q_0$ whose trajectories remain in $Q_0$, so the behavior of $f$ outside $Q$ does not concern us. To view $f$ as a map from $Q$ into the plane, we can think of $f$ as a map that involves linear horizontal contraction of $Q$, linear vertical

Figure 3. The horseshoe map $f$ is obtained by horizontally squeezing the rectangle in (a) to obtain (b); then vertically stretching it to obtain (c); then folding it to obtain (d). The images under $f$ of the points $A$, $B$, $C$, $D$ are denoted by $A^*$, $B^*$, $C^*$, $D^*$, respectively. The set $f(Q) := \{x \in Q : f(x) \in Q\}$ is the union of two horizontal strips, $S_1$ and $S_2$. They are mapped onto the two vertical strips of $Q \cap f(Q)$. May 2001]
stretching of $Q$, and finally folding the stretched and contracted $Q$ once before placing it across itself as indicated in Figure 3a. For a more detailed, but elementary, discussion of this map, see [1]. For a sampling of the large literature on such horseshoes, see [8].

**Definition.** A sequence $S := (S_{i_0}, S_{i_1}, \ldots, S_{i_n}, \ldots)$ of symbol sets is called a *forward itinerary* (in $\{S_1, S_2\}$). We say a point $x$ follows $S$ if $f^n(x) \in S_{i_n}$ for all $n = 0, 1, 2, \ldots$.

**Proposition 1. (The Horseshoe.)** Let $S_1$ and $S_2$ be as described above, and let $f$ be the horseshoe map. Then all forward itineraries in $\{S_1, S_2\}$ are followed.

The proof illustrates the technique we use throughout the paper: We introduce the terms “expander” and “family of expanders” in the following definition; the family of vertical curves discussed in the proof is a family of expanders for $\{S_1, S_2\}$.

By a *vertical curve* we mean the graph in the plane of a continuous function $x = v(y)$. We define the family of sets

$$E := \{ E : E \text{ is a vertical curve in } Q \text{ connecting the upper and lower sides of } Q \}.$$ 

Note the following two properties:

(F1) $E$ is a nonempty family of nonempty subsets of $Q$, and

(F2) for each $E \in E$, and each $S_i$ there is a compact set $P_i \subset E \cap S_i$ such that $f(P_i) \in E$, that is, $f(P_i)$ “expands” $P_i$ to a member of $E$.

(Actually, in this case we can choose $P_i := E \cap S_i$.)

**Definition.** Whenever $E$ is a set of sets satisfying (F1) and (F2), we call $E$ a *family of expanders* for $\{S_1, S_2\}$. The vertical curves $E \in E$ are called *expanders*, and any compact subset $P$ of an expander is called a *pre-expander* if $f(P) \in E$.

The proof of Proposition 1 uses the Covering Principle, which is given after that proof.

**Definition.** We say a set $P$ covers $E$ for $f$ if $f(P) \supseteq E$. This term is motivated by asking when the equation $f(p) = e$ has a solution $p \in P$ for every $e \in E$. The answer, of course, is that $P$ must cover $E$. If, for example, $A$ is an $n \times n$ matrix, we can solve $Ap = e$ for each $e \in E$ if $A$ covers $R^n$.

**Proof of Proposition 1.** Now, let $S = (S_{i_0}, S_{i_1}, \ldots, S_{i_n}, \ldots)$ be any forward itinerary. Let $E_0$ be any expander; property (F2) ensures that $E_0$ contains a pre-expander $P_0 \subset S_{i_0}$ (namely $E_0 \cap S_{i_0}$). Then $f(P_0)$ is an expander, which we denote by $E_1$. Now we repeat the procedure: The expander $E_1$ contains a pre-expander $P_1 \subset S_{i_1}$. Then $f(P_1) = E_2$ is an expander. Proceeding in this way we obtain the sequence of nonempty, compact sets $P_n \subset E_n$, where $E_{n+1} := f(P_n)$ for all positive integers $n$. Since $P_n \subset S_{i_n}$, it follows from the following Covering Principle that $S$ is followed by some $x_0$.

**Proposition 2. (The Covering Principle.)** Let $Q$ be a metric space. Assume that $f : Q_0 \to Q$ is continuous, where $Q_0 \subset Q$ is compact. Let

$$P_0, P_1 \subset E_1, P_2 \subset E_2, \ldots, P_n \subset E_n, \ldots$$
be nonempty, compact subsets of \( Q_0 \) such that \( f(P_n) \supset E_{n+1} \) for each positive integer \( n \). Then there exists a point \( x_0 \in P_0 \) that is mapped indefinitely, i.e., \( x_n := f^n(x_0) \) is defined for all \( n \), and \( x_n \in P_n \).

Proof. Write \( P_i^j \) for \( P_i \) for all \( i \). We construct families of compact sets \( P_i^{j+1} \subset P_i^j \) for \( j = 1, 2, \ldots \) by setting \( P_i^{j+1} := \{ x \in P_i^j : f(x) \in P_i^{j+1} \} \). Hence \( f(P_i^{j+1}) = P_i^j \). Let \( P_i^\infty \) be the nonempty compact set \( \cap_{j=1}^{\infty} P_i^j \). It follows that \( f(P_i^\infty) = P_i^{\infty+1} \), so for any \( x_0 \in P_0^\infty \), \( f^n(x_0) \) is defined and is in \( P_n \).

This proof is quite different from other proofs of topological horseshoe theorems, and generalizes arguments in [6]. Other mathematicians who have investigated horseshoes without hyperbolicity include A. Szymczak [10], K. Burns and H. Weiss [2], and K. Mischaikow and various coauthors (see [4], [3], and [7]). Their proofs are based on some form of topological degree argument such as the Conley index. Our approach is an extension of the construction of Cantor sets and is an extension of arguments often used in demonstrating chaos for one-dimensional maps, e.g., in [5]. Approaches via topological degree theory require assuming the existence of an isolating neighborhood; ours does not.

A minor variant of the proof given of Proposition 1 illustrates that it is more important that \( E \) satisfy (F1) and (F2) than it is to worry about what sets are in \( E \). Define \( E_{con} := \{ E \subset Q : E \text{ is compact and has a connected component that intersects both the bottom } AB \text{ and the top } CD \text{ of } Q \} \).

Another proof of Proposition 1. We use \( E_{con} \) in the above proof. Here again \( P_n \) can be chosen to be \( E_n \cap S_{in} \).

Figure 4. BULGING HORSESHOE. The map \( f \) shown here is non-invertible. Again \( S_1 \cup S_2 = f(Q) \), but now, because of the bulge, \( S_2 \) has two connected components (b). In (c) a possible \( E \in E \) is shown along with its image. Let \( E_i = f(E \cap S_i) \) for \( i = 1, 2 \). While \( E \cap S_2 \) is not connected, its image \( E_2 \) is.
The Expander Lemma is applicable in many other situations for an appropriately chosen family of expanders. Consider, for example, the “Bulging Horseshoe” (Figure 4), the “Pig’s Tail” (Figure 5), and the “Mutated Horseshoe” (Figure 6). We can take the same symbol sets $S_1$ and $S_2$ for these examples, and apply the argument we used for the Smale Horseshoe map. We encapsulate the argument in Lemma 3.

Smale recognized that if $f(Q)$ crosses $Q$ from top to bottom $K$ times, there must be a trajectory following each sequence of $K$ symbol sets; see Figure 7. For this case...
also, all itineraries are followed but with three symbol sets. Again the horizontal strips that map into \( Q \) serve as symbol sets, while vertical curves serve as expanders.

3. FAMILIES OF EXPANDERS. In the examples of the previous section, the role played by the families \( \mathcal{E} \) and \( \mathcal{E}_{\text{con}} \) are fundamental in proving the existence of chaos. They act as sensors of expansion in the dynamics (hence the name “expanders”) and they have specialized subsensors located in all symbol sets (that is, subsets lying in the symbol sets) that, when mapped, become expanders (which is why we call them “pre-expanders”). This sounds like a vicious circle, but it isn’t! On the contrary, it works surprisingly well. The definition of “expander” varies from problem to problem; what is constant is how they relate to each other in “families”.

The basic setting for the remaining results of this paper is the following:

**Hypothesis H.** Let \( Q \) be a metric space, let \( Q_0 \subset Q \) a compact subset, let \( S_1, S_2, \ldots, S_k \) be nonempty, compact subsets of \( Q_0 \), with \( K \geq 2 \), and assume \( f: Q_0 \rightarrow Q \) is a continuous map.

In the next section we add the hypothesis that the \( S_i \) are pairwise disjoint, but there are interesting cases where they are not.

**Definition.** Assume Hypothesis H. Let \( \mathcal{E} \) be a collection of subsets of \( Q \) such that

(F1) \( \mathcal{E} \) is a nonempty family of nonempty subsets of \( Q \), and

(F2) for each \( E \in \mathcal{E} \) and each \( S_i \), there are compact sets \( P_i \subset E \cap S_i \) such that \( f(P_i) \in \mathcal{E} \).

Then we say that \( \mathcal{E} \) is a family of expanders and each member \( E \in \mathcal{E} \) is an expander (for \( f \) and \( \{S_1, S_2, \ldots, S_k\} \)).

Now we are ready for the Expander Lemma.

**Lemma 3. (The Expander Lemma.)** Assume Hypothesis H and let \( \mathcal{E} \) be an associated family of expanders. For each sequence \( S := (S_{in})_{n>0} \) of symbol sets there exists a point \( x_0 \in S_{i_0} \) that follows \( S \).

**Proof.** The proof is the same as the proof of the Horseshoe Proposition from the beginning of the second paragraph on.

4. SENSITIVE DEPENDENCE ON INITIAL DATA. If we assume only Hypothesis H, then we could have a trivial situation in which all \( S_i \) are equal. If \( \cap S_i \) contains a fixed point \( x \), i.e., \( f(x) = x \), then \( x \) trivially follows every sequence \( S \) since \( f^n(x) \in S_{i_n} \). Under the following pairwise disjointedness Hypothesis PD, the situation is quite interesting.

We write \( \text{dist}(x, y) \) for the distance between \( x \) and \( y \) in \( Q \). Often we require the following additional assumption:

**Hypothesis PD. (Pairwise Disjoint)** Assume Hypothesis H, and assume additionally that \( S_1, \ldots, S_k \) are pairwise disjoint sets. Let \( \sigma \) denote the minimum distance between these sets, i.e., \( \sigma = \min \text{dist}(x, y) \), where the minimum is taken over all \( x \) and \( y \) that are in different symbol sets.

We investigate some cases where every forward itinerary is followed; that is, every forward itinerary has a point that follows it. In most of our examples \( f \) is defined on

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a larger sets, but we are concerned only with trajectories lying entirely in \( Q_0 \). Loosely speaking, the map \( f \) is “chaotic” if it possesses the following sensitivity property: There is a compact set \( Q_{\text{chaos}} \subset Q_0 \) that is invariant, i.e., \( f(Q_{\text{chaos}}) = Q_{\text{chaos}} \), and for every itinerary \( S \) there is a “sensitive” point \( x \in Q_{\text{chaos}} \) that follows \( S \). See Lemma 4 for more detail.

This definition suggests the game of pinball, where bumpers or reflectors forcefully repel a ball that hits them. We might label these bumpers \( S_1, \ldots, S_k \). For some pinball games, once the ball is launched, it could conceivably bounce forever between the bumpers if perfectly positioned, and the ball could collide with the bumpers in any sequence. It could even collide with only a single bumper (bouncing off the wall and back to the bumper). It is much more likely that after a few bounces, the ball falls into a hole that ends play. This rich pattern of possibilities makes the game fun and justifies calling its dynamics “chaotic”.

**Definition.** We say a compact set \( Q_* \subset Q_0 \) is invariant if \( f \) is defined at all points of \( Q_* \), and \( f(Q_*) = Q_* \). Let \( Q_* \subset Q_0 \) be a nonempty compact invariant set. If \( x \in Q_* \), \( f^n(x) \) is defined for all positive integers \( n \). Let \( x \in Q_* \) and \( x_i \in Q_* \) for \( i \) a positive integer. We say the sequence \( \{x_i\} \) separates from \( x \) (or more precisely, the trajectories of \( x_i \) separate from the trajectory of \( x \)) if \( x_i \to x \) as \( i \to \infty \) and there is a \( \delta > 0 \) such that for all \( i \) there is a positive integer \( m = m(i) \) such that \( \text{dist}(f^m(x_i), f^m(x)) > \delta \) for all \( i \). An \( x \) having such a sequence with all \( x_i \) in \( Q_* \) is called sensitive to initial data (in \( Q_* \)). We say a set \( Q_* \) is chaotic if it is nonempty and invariant, and every \( x \) in \( Q_* \) is called sensitive to initial data in \( Q_* \).

In practice we assume PD and have \( x_i \to x \) as \( i \to \infty \) where \( x \) and each \( x_i \) follow itineraries \( S \) and \( S_i \), respectively. Furthermore, we assume \( S \) is not equal to any of the \( S_i \). Hence, for each \( i \), there is a time \( m \) such that \( f^m(x) \) and \( f^m(x_i) \) are in different symbol sets. Hence, \( \text{dist}(f^m(x_i), f^m(x)) > \sigma \).

**Lemma 4. (The Chaos Lemma.)** Assume Hypothesis PD. Then there is a chaotic set \( Q_* \subset Q_0 \) such that each forward itinerary is followed by some point in \( Q_* \).

**Outline of Proof.** Lemma 4 implies that there is a chaotic invariant set \( Q_* \). Rather than give a detailed proof, we outline the proof and challenge the reader to complete each step.

1. **(Invariance)** If \( x \in Q \) and if \( f^m(x) \) is defined and in \( Q \) for all positive integers \( n \), then define

   \[
   \omega(x) := \{ z : z \text{ is a limit point of } \{ f^m(x) \} \}.
   \]

   Then \( \omega(x) \) is a compact invariant set.

2. **(Following)** Let \( S^* \) be an itinerary with the property that every finite sequence \( (S'_i)_{i=1}^m \) occurs in \( S^* \) in order; that is, for some \( j \), the \( j+1 \) through \( j + m \) terms in \( S^* \) are equal to \( S'_i \) through \( S'_{i+m} \), respectively. Let \( x_{(1)} \) be a point that follows \( S^* \). Then for each itinerary there is a point \( x_{(2)} \in \omega(x_{(1)}) \) that follows it. In particular, some \( x_{(2)} \in \omega(x_{(1)}) \) follows \( S^* \). Notice that \( \omega(x_{(2)}) \subset \omega(x_{(1)}) \).

3. **(Minimality)** There is a point \( x^* \) in \( Q_0 \) which follows \( S^* \) and also satisfies \( x^* \in \omega(x^*) \). Write \( Q^* := \omega(x^*) \). **Hint:** Consider the set \( \mathcal{L} \) of all compact sets \( L \) equal to \( \omega(x) \) for a point \( x \) that follows \( S^* \). Show that there is a smallest \( L \), that is, an \( L \) that contains no other set in \( \mathcal{L} \).
4. (Sensitivity) Write \( x_i^* \) for \( f^i(x^*) \) for positive integer \( i \). Then for \( i \neq j \), the itineraries of \( x_i^* \) and \( x_j^* \) are different. Hence, if some subsequence \( x_{i_n}^* \) converges to \( y \), then \( y \) is sensitive to initial data in \( Q^* \). But every \( y \) in \( Q^* \) has such a sequence.

5. ONE-SIDED TRAJECTORIES IMPLY DOUBLE-SIDED TRAJECTORIES.
We have so far considered only one-sided trajectories, but in the setting of all the lemmas, double-sided trajectories exist. Actually, this is a direct consequence of existence of one-sided trajectories. We now formulate and prove it. Let \( \mathbf{Z} \) denote the set of integers.

**Proposition 5.** Assume Hypothesis H. Let \( Q^* \subset Q \) be a compact invariant set. Assume that for every one-sided sequence \( (S_{i_n})_{n=0}^\infty \) of symbol sets, there is a point \( x_0 \in Q^* \) that follows \( S \). Then for every double-sided sequence \( \mathcal{D} := (S_{i_n})_{n \in \mathbf{Z}} \) of symbol sets, there exists a sequence \( (x_n)_{n \in \mathbf{Z}} \) in \( Q^* \) such that \( x_n \in S_{i_n} \) and \( f(x_n) = x_{n+1} \), for all \( n \in \mathbf{Z} \).

**Proof.** Given any double-sided symbol sequence \( \mathcal{D} := (S_{i_n})_{n \in \mathbf{Z}} \) we consider the truncated, one-sided symbol sequence \( S_{-n} := (S_{i_n})_{j=-n}^\infty \). By assumption there is a point \( y_{-n} \in Q^* \) that follows \( S_{-n} \). (Of course, \( y_{-n} \in S_{i_{-n}} \).) Next define \( z_{n} := f^{n}(y_{-n}) \in S_{i_{n}} \). The sequence \( (z_n)_{n \in \mathbf{Z}} \) has a subsequence \( (z_{n_j}) \) that converges to a point \( x_0 \in S_{i_0} \cap Q^* \). Since each point \( z_{n_j} \) follows \( D_{0} \), the point \( x_0 \) also follows \( D_{0} \) by continuity. Set \( x_m := f^m(x_0) \) for \( m \) a positive integer.

Our goal is to choose a subsequence \( (n_k) \) such that the trajectories with initial points \( y_{-n_k} \) converge to a trajectory defined for all time. Define \( z_{m,n} := f^{n+m}(y_{-n}) \in S_{i_{m}} \) for each \( m \) with \( n + m \geq 0 \). For fixed \( m \in \mathbf{Z} \) there is a sequence \( n_i \rightarrow \infty \) such that the sequence \( z_{m,n_i} \) converges to a point \( x_{m} \in S_{i_{m}} \cap Q^* \), and its trajectory follows \( S_{m} \). Using a diagonalization argument, the sequence \( (n_i) \) can be chosen independent of \( m \).

By continuity \( x_{m+1} = f(x_m) \) and \( x_m \in S_{i_m} \cap Q^* \).

6. SUBSHIFTS OF FINITE TYPE. The Chaos Lemma can be extended to the “subshift of finite type” case where not all of the subsets \( S_i, i = 1, 2, \ldots, K \), are interrelated by the map \( f \). To define a restricted interrelationship, consider a transition matrix \( M = (m_{ij})_{1 \leq i, j \leq K} \); that is, each entry \( m_{ij} \) of \( M \) is either 0 or 1.

**Definition.** Let \( S := (S_{i_n}) \) be an itinerary (either one-sided or two-sided). We say that \( S \) is \( M \)-admissible if \( m_{ij} = 1 \) whenever \( S_i \) and \( S_j \) are consecutive terms in \( S \) with \( S_i \) preceding \( S_j \).

So far \( M \) could be incredibly dull, say if \( m_{ij} = 0 \) for all \( i \) and \( j \), or if \( m_{ij} = 1 \) if and only if \( i = j \). The following ensures interesting dynamics.

**Hypothesis M.** Suppose that \( M = (m_{ij})_{1 \leq i, j \leq K} \) is a transition matrix. There is an \( M \)-admissible sequence \( S := (S_{i_n}) \) in which each of the sets \( S_j \) occurs for infinitely many \( n \) and for at least one \( i \), there are \( j \) and \( k \) with \( j \neq k \) such that \( m_{ik} = 1 = m_{ij} \).

**Definition.** Assume Hypothesis H and let \( M = (m_{ij})_{1 \leq i, j \leq K} \) be a transition matrix. Let \( \mathcal{E} \) be a nonempty collection of nonempty compact subsets of \( Q \). We say that the map \( f \) has a set \( \mathcal{E} \) of expanders for \( \{S_1, S_2, \ldots, S_K\} \) and \( M \) if the following two conditions hold:

1. For every \( i = 1, 2, \ldots, K \) there exists an \( E \in \mathcal{E} \) with \( E \cap S_i \neq \emptyset \).

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2. If $E \cap S_i \neq \emptyset$ for an $E \in E$, and $m_{ij} = 1$, then there is a compact set $P_{ij} \subset E \cap S_j$ such that $f(P_{ij}) \cap S_j \neq \emptyset$.

With the more general definition of a set of expanders, we can now formulate a more general version of the Chaos Lemma.

**Lemma 6. (Chaos Lemma for Subshifts of Finite Type.)** Assume Hypothesis M. If the map $f$ possesses an expander set $E$ for $\{S_1, S_2, \ldots, S_k\}$ and $M$, then there is a chaotic set $Q_* \subset Q_0$ such that for every $M$-admissible sequence $S$, there is a point $x \in Q_*$ such that $x$ follows $S$.

**Proof.** Combine and adapt the arguments of Lemmas 3 and 4.

As a simple and nice example, consider a quadratic map $f : [0, 1] \to [0, 1]$ with a period-three orbit $x_1, x_2, x_3$, where $x_2 = f(x_1)$, and $x_3 = f(x_2)$, and $x_1 < x_2 < x_3$. If we choose two symbol sets, $S_1 = \{x_1, x_2\}$ and $S_2 = \{x_2, x_3\}$, then $f(S_1) \supset S_2$, $f(S_2) \supset S_1$, and $f(S_2) \supset S_2$. By setting $E := \{S_1, S_2\}$ and $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, it follows from Lemma 6 that for any sequence $S = (S_{in})$ of symbol sets in which no pair of consecutive $S_{in}$ are both $S_2$, there is a point $x_0 \in S_{i_0}$ that follows $S$. In this case $E$ is finite and the symbol sets are the expanders.

7. **WHEN DOES $Q$ CONTAIN AN INVARIANT SET?** Mathematics courses for mathematics majors or graduate students present pearls of wisdom. Exercises stress the ability to prove results rather than how to find formulations, a frequently difficult art. For the mathematician, a proof is only half the battle of creating a new theorem. Before the proof must come the discovery of a new statement or conjecture. Students are rarely asked to come up with theorems or conjectures that are new to them. In this section, we do not know what the final conjecture should be, but we have some examples and ideas. While mathematicians do not have a widely accepted term for “pseudo-conjecture”, i.e., a statement that gives only the general form of the conjecture, it is precisely this process of creating that proto-mathematicians must master.

Until now we have avoided the case $K = 1$ because it seems a bit trivial: there would be only one possible itinerary, the constant itinerary $S_1$. But saying that the itinerary is followed implies that there is an invariant set in $Q$ (and more specifically in $S_1$), so the case $K = 1$ has a non-trivial conclusion. We have generalized the notion of horseshoe maps in this paper, but further generalizations could be possible if the case $K = 1$ were better understood. Figure 8 shows it is necessary to be careful.

**Figure 8.** The set $f(Q)$ is obtained by rotating $Q$ by 90°, with that rotation about a point not in $Q$. In this case there is no fixed point in $Q$. Furthermore, if $x \in Q$, then $f^2(x) \notin Q$, so $Q$ contains no nonempty invariant set.
When does $f$ have an invariant set $Q$? We define $S_1$ to be \( \{ x \in Q : f(x) \in Q \} \). The simplest case is $f(Q) \subset Q$, so the family of expanders can be chosen to be all nonempty compact sets. The Expander Lemma says that all sequences are followed (extended to the $K = 1$ case), so there is a compact invariant set, following the arguments of the Chaos Lemma (even though there is no chaos guaranteed). It might contain just one point, a fixed point, and it might not contain a fixed point. Now suppose $S_1$ is a proper subset of $Q$. In the two-dimensional example our sets $E$ in some sense connect the top to the bottom of $Q$, that is, $CD$ to $AB$. Figure 9 is such an example.

![Figure 9.](image)

**Pseudo-Conjecture.** If for each set $E$ that “stretches across” $Q$ the image $f(E) \cap Q$ also stretches across $Q$, then there is a compact, nonempty invariant set in $Q$.

We do not know how “stretches across” should best be defined, but we believe Figures 10 and 11 are examples that could be covered. We conjecture that these examples have invariant sets. In these $f(Q) \cap Q$, the top of $Q$ is connected to its bottom indirectly via some kind of linking of components of $f(Q) \cap Q$. We hope some reader(s) can show that Figures 10 and 11 contain invariant sets. If so, it is probably a small step to show that Figure 12 has chaos with $K = 2$.

![Figure 10.](image)

**REFERENCES**


Figure 11. The three pieces of $f(Q) \cap Q$ are parts of the three Borromean rings. The circle, denoted $C$, is not the boundary of a disk $Q \setminus f(Q)$, but is the boundary of a manifold (and so is homologically trivial, but not homotopically trivial). The boundary of $Q$ contains top $T$, bottom $B$, and side $S$. We conjecture that if $f(Q)$ does not intersect $S$ but $C$ is not homotopically trivial in $Q \setminus f(Q)$, then there is an invariant set in $Q$ and there is a family of expanders with $K = 1$.

Figure 12. We conjecture that this example has a family of expanders for $K = 2$. We would choose $S_1$ and $S_2$ as indicated by their images.

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**Mathematical Constance**

by Arthur Benjamin (with apologies to Joyce Kilmer)

I think that I shall never see
A constant lovelier than $e$.
Whose digits are too great to state;
They’re 2.71828 . . .
And $e$ has such amazing features.
It’s loved by all (but mostly teachers).
And Calculus, we’d not do well in
Without such terms as $exp$ and $ln$.
For $e$ has such nice properties,
Most integrals are done with . . . ease.
Theorems are proved by fools like me,
But only Euler could make an $e$.

I guess, though, if I had to try
To choose another constant, I
Might offer $i$ or $\phi$ or $\pi$,
But none of those would satisfy.
Of all the constants I know well,
There’s only one that rings the Bell.
Not $\pi$, not $i$, nor even $e$.
In fact, my Constance is a she.
It’s Constance Reid. I wouldn’t fool ya’.
With books like *Hilbert*, *Courant*, and *Julia*,
Of all the Constance you will need,
There’s only one that you should Reid.