Testing Whether Two Chaotic One Dimensional Processes Are Dynamically Identical

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Consider the situation where two individuals observe the same chaotic physical process but through time series of different measured variables (e.g., one individual measures a temperature and the other measures a voltage). If the two individuals now use their data to reconstruct (e.g., via delay coordinates) a map, the maps they obtain may appear quite different. In the case where the resulting maps appear one dimensional, we introduce a method to test consistency with the hypothesis that they represent the same physical process. We illustrate this method using experimental data from an electric circuit.

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In the case of chaotic physical processes, in many situations it has proven possible, via examination of output data only, to obtain a reconstruction of a dynamics corresponding to the observed system [1]. The selection of outputs, of course, is quite arbitrary. A natural question that arises is how two observers, who use their own choices of outputs, are able to tell whether they are studying the same dynamical process.

In this paper we present a method that can be used to test consistency with the hypothesis that two one dimensional maps originate from the same physical process. We imagine that these maps are obtained by, for example, using time-delay embedding [1] or by recording consecutive maxima in the construction of the dynamics from the data is by recording to this total scatter as noise. Another way to obtain a regression slightly greater than 2). For convenience, we refer partly dynamical (due to the fact that the attractor dimension is slightly greater than 2). We call g the canonical conjugate of f. That two one dimensional maps have identical canonical conjugates would suggest that they originate from the same physical process [4].

In Fig. 3(b) we plot the map defined by $f_x: x_n \mapsto x_{n+1}$. This map looks quite different from the map in Fig. 3(a). We address the following question: If one is presented with the plots in Figs. 3(a) and 3(b), can one test to determine consistency with the hypothesis that the two represent different reconstructions obtained from the same chaotic process?

Let $A = [c, d]$ be a chaotic one dimensional attractor on which the dynamics is given by a continuous map $x_{n+1} = f(x_n)$ for $x_n$ in $A$ with the following properties: (i) $f$ admits a natural measure $\mu$ (i.e., the measure of an open interval $B$ is the fraction of iterates that a typical trajectory spends in $B$); (ii) any subinterval of $[c, d]$ has positive natural measure [3]. Using the natural measure $\mu$, we define a change of coordinate $h: [c, d] \rightarrow [0, 1]$ by letting $w_n := h(x_n) := \mu([c, x_n])$, i.e., $w_n$ is equal to the natural measure to the left of $x_n$. In terms of $w_n$, the map $f$ is transformed to a new map $g: [0, 1] \rightarrow [0, 1]$ given by $g(w_n) := h \circ f \circ h^{-1}(w_n)$. [A similar construction is via the change of variable $\tilde{h}(x_n) := \mu([x_n, d])$. This will result in a map which is the same as $g$ except for flipping over each axis once. We do not distinguish between the two.] We call $g$ the canonical conjugate of $f$. That two one dimensional maps have identical canonical conjugates would suggest that they originate from the same physical process [4].

Notice that, by construction, the map $g(w_n)$ generates chaotic trajectories whose natural measure corresponds to a uniform density for $w_n$ in $[0, 1]$. Since $f$ and $g$ differ

FIG. 1. The attractor observed in our experiment.
only by a change of coordinate, the topological dynamics of \( f \) and \( g \) are the same. In particular, for every periodic orbit of \( f \) there is a corresponding periodic point of \( g \) with the same period. Most importantly, \( f \) and \( g \) have the same Lyapunov exponent (with respect to their natural measure); this is easy to see if \( h \) is piecewise smooth, but it is also suggested in greater generality by Pesin’s formula [5], which asserts that (under some technical conditions) the Lyapunov exponent is equal to the metric entropy. Therefore, if two one dimensional maps \( f_1 \) and \( f_2 \) with canonical conjugates \( g_1 \) and \( g_2 \) are dynamically different, their difference can be evaluated by comparing \( g_1 \) with \( g_2 \). This will be discussed later in more detail [see Eq. (1)].

In physical experiments noise and measurement errors are unavoidable; thus two maps that are close together cannot be distinguished by our method. A more serious challenge is due to the fact that trajectories that appear to be chaotic may eventually settle on a periodic attractor, as is the case for the logistic maps at certain parameter values. As a result, one cannot rigorously check whether a canonical conjugate is meaningful. Nonetheless, in many situations, experimental data are highly consistent with the assumption of the existence of a chaotic one dimensional attractor that satisfies our two conditions; therefore we expect that our method can be applied to a wide class of experiments. To demonstrate how this is possible, in the following we present results of applying the canonical conjugate method to a simple experiment, where both noise and measurement errors are present.

As an example, we compare the canonical conjugates of \( f_y \) and \( f_x \) of Figs. 3(a) and 3(b). We first calculate the canonical conjugate of \( f_y \), denoted by \( g_y \). This can be obtained by the following steps. First, rescale the data with respect to the natural measure. That is, associate to \( y \) a new coordinate value \( w \), where \( w \) represents the natural measure (fraction of data points) to the left of \( y \). Next, replot the map in terms of the new variable \( w \) (not shown). Finally, fit the plot with a piecewise linear map. The curve in Fig. 4 shows \( g_y \) obtained by piecewise linear fitting, for which we have applied the constraint \( 1/|g'_y(w_1)| + 1/|g'_y(w_2)| = 1 \) for all \( w_1 \) and \( w_2 \) for which \( g_y(w_1) = g_y(w_2) \). This constraint is known as the Perron-Frobenius equality, which is the necessary and sufficient condition for the density generated by \( g_y(w) \) to be constant. The Lyapunov exponent of \( g_y \) is simply \( \lambda_y = \int_0^1 \log|g'_y(w)| \, dw \). Using this, we get \( \lambda_y = 0.42 \).

Although there is no simple formula that associates the sequence \( \{y_n\} \) (obtained from the \( x = 0 \) surface of section)
with \( \{x_n\} \) [the successive maxima of \( x(t) \)], either map can be obtained from the other by a coordinate change. We compute the canonical conjugate of \( f_x \), denoted by \( g_x \), using the same algorithm. It agrees so well with \( g_x \) that each curve falls into the thickness of the other; therefore, we do not show \( g_x \) separately. This apparent similarity between \( g_x \) and \( g_y \) suggests that \( f_y \) and \( f_x \) correspond to the same dynamical process.

In general, a different surface of section will lead to a different Poincaré return map. We have arbitrarily chosen several different surfaces and find good agreement between their associated canonical conjugates.

As another example, Fig. 5 shows the cross section of the attractor in the \( x-z \) plane at \( y = 1 \). In this case, our procedure to evaluate the return map needs to be modified. In particular, we first perform a change of coordinates to unfold the attractor; the specific choice of this change of coordinate is somewhat arbitrary. As an example of one such unfolding, we choose the polar coordinate \((r, \theta)\) with the origin at the point \((x_0, z_0) = (2.5, 0.6)\). In terms of \( \theta \) the return map is well defined (i.e., considering the cross section of the attractor shown in Fig. 5(a) is one dimensional, there is a one-to-one relationship between points in the cross section and values of \( \theta \)) and is shown in Fig. 5(b). Comparing Fig. 5(b) with Fig. 3(a), we find no noticeable similarities. However, the canonical conjugate \( g_\theta \) agrees well with \( g_y \). In fact, our plot of \( g_\theta \) also falls into the thickness of the line plotting \( g_y \) in Fig. 4.

Finally, we investigate a situation where the return map is one dimensional but discontinuous. Similarly to Fig. 3(b), we generate a sequence \( \{z_n\} \) by recording consecutive peaks in the \( z \) coordinate. We then define a map \( f_z: z_n \mapsto z_{n+1} \). Figure 6(a) shows that \( f_z \) is discontinuous. The reason is that in our experiment certain maxima are missed when the signals are below a detectable threshold. To understand what is happening

![Diagram](image-url)
It follows immediately that \( d(g_1, g_2) \geq |\lambda(g_1) - \lambda(g_2)| \), where \( \lambda(g_i), i = 1, 2 \), is the Lyapunov exponent of \( g_i \). On the other hand, two maps that have the same Lyapunov exponent may not be dynamically similar and the distance between them may not be small. For the examples discussed in this paper, we find \( g_y \), \( g_x \), and \( g_\theta \), which are derived from the same circuit, are within 0.07 of each other with respect to this metric, whereas \( d(g_p, g_y) \) is approximately equal to 0.28, which is considerably larger even though their Lyapunov exponents are almost equal \[ |\lambda(g_p) - 0.41 \pm 0.02|, \quad |\lambda(g_y) - 0.43 \pm 0.02| \].

In conclusion, we have introduced a method whereby, given two one dimensional chaotic maps, we can compute and compare their canonical conjugates to check if they might represent the same physical process, i.e., if they have the same dynamical properties. We have demonstrated that this method is robust in that it can be implemented in a laboratory experiment in which noise is present.

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[3] If the attractor cycles through \( m \) component intervals disconnected from each other, then we can apply our construction by considering the action of \( f^m \) on one of the component intervals. Although not mathematically proven, it is commonly anticipated that chaotic attractors in one dimensional maps typically consist of a finite number (often one) of component intervals.

[4] One can also test whether two one dimensional maps \( x_{n+1} = f_1(x_n), \ y_{n+1} = f_2(y_n) \) originate from the same physical process without using the canonical conjugates. In particular, we consider the following coordinate change \( y := h_2^{-1} \circ h_1(x) \), where \( h_1(x) \) [respectively, \( h_2(y) \)] represents for \( f_1(x) \) [respectively, \( f_2(y) \)] the natural measure to the left of the point \( x \) (respectively, \( y \)). Thus the map \( f_1(x) \) is transformed to a new map \( f_2(y) \). That \( f_1 \) and \( f_2 \) are identical suggests that \( f_1 \) and \( f_2 \) represent the same physical process.


[6] The dynamics of the circuit can be approximately described by a set of three ordinary differential equations. The third equation is \( dz/dt = f(x) - z \), where \( f(x) = 0 \) for \( x < 3 \) and \( f(x) = k(x - 3) \) for \( x \geq 3 \) [2]. For the circuit that produced \( g_y \), \( g_x \), and \( g_\theta \), \( k \) was chosen as \( k = 15 \). For the circuit that produced \( g_p \), \( k \) was changed to \( k = 10 \).