Explosions of chaotic sets

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Abstract

Large-scale invariant sets such as chaotic attractors undergo bifurcations as a parameter is varied. These bifurcations include sudden changes in the size and/or type of the set. An explosion is a bifurcation in which new recurrent points suddenly appear at a non-zero distance from any pre-existing recurrent points. We discuss the following. In a generic one-parameter family of dissipative invertible maps of the plane there are only four known mechanisms through which an explosion can occur: (1) a saddle-node bifurcation isolated from other recurrent points, (2) a saddle-node bifurcation embedded in the set of recurrent points, (3) outer homoclinic tangencies, and (4) outer heteroclinic tangencies. (The term “outer tangency” refers to a particular configuration of the stable and unstable manifolds at tangency.) In particular, we examine different types of tangencies of stable and unstable manifolds from orbits of pre-existing invariant sets. This leads to a general theory that unites phenomena such as crises, basin boundary metamorphoses, explosions of chaotic saddles, etc. We illustrate this theory with numerical examples. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Many papers have been written concerning global bifurcations involving observable phenomena such as jumps in chaotic attractors (crises) [1,2] and sudden changes in basin boundaries (metamorphoses) [3–6]. These phenomena have also been observed in experiments including those done for a convect-

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1 The term metamorphosis has been used to denote a sudden jump in the basin boundary.

ing superfluid [7], modulated CO 2 [8–10] and NMR [11–14] lasers, a spin wave system [15–18], plasma discharge [19], a magnetically driven mechanical system [20–22], Josephson junctions [23,24], a nonlinear microwave resonator [25], electronic circuits [26–29], driven semiconductor oscillators [30], a driven nonlinear p–n junction [31], and a leaky faucet [32–34].

Needless to say, for the experimentalist, it is of great importance to know if any large deviation due to a change of parameter or background noise, occurs in his or her system. A better understanding of typical global bifurcations is therefore required. The goal of this paper is to describe a theory which underlies all
Fig. 1. Explosion diagrams: (a) An “explosion” of recurrent points at \((\mu_c, \tilde{y})\) as \(\mu\) increases. The point \(\tilde{y}\) may or may not be recurrent, but \((\mu_c, \tilde{y})\) must be the limit of points \((\mu, y)\) for which \(y\) is recurrent; (b) Same as (a) except that explosion occurring as \(\mu\) decreases.

For example, if there is a generic saddle-node bifurcation at \((\mu_c, \tilde{y})\) and there are no other recurrent point near \((\mu_c, \tilde{y})\), then it is an explosion point. On the other hand, a generic period doubling at \((\mu_c, \tilde{y})\) is not an explosion point, because there are recurrent points arbitrarily near \((\mu_c, \tilde{y})\) both for \(\mu < \mu_c\) and for \(\mu > \mu_c\) (the new recurrent period 2 points converge to the original period 1 orbit as \(\mu\) approaches \(\mu_c\)).

When there is sufficient hyperbolicity, a dynamical system is structurally stable [42], i.e., changes in a parameter can result in no new dynamics, and thus there are no explosions. The early motivation for studying explosions in Smale [43] was based on the goal of finding many structurally stable dynamical systems. Our motivation, however, is to describe what happens in common physical systems when structural stability fails. This failure occurs when stable and unstable directions cannot be distinguished, such as at tangencies of stable and unstable manifolds of fixed or periodic points or at local bifurcations of these orbits. As mentioned previously, saddle nodes which are isolated from other recurrent points are explosions, while period doubling bifurcations are not.

The set of recurrent points naturally decomposes into blocks called “basic sets”. A basic set is a compact invariant set with a dense orbit that is not contained in a larger set of this kind.\(^2\) Some examples of sets that could be basic sets are periodic orbits, chaotic

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\(^2\)An isolated periodic orbit is a basic set, as is a chaotic attractor or a chaotic saddle, but a periodic orbit embedded in a chaotic set is not a basic set because it lies in a larger set that is a basic set.
attractors, and chaotic saddles (unstable invariant sets with dense orbits such as the invariant set of a horseshoe). We observe that new basic sets begin with a fixed point or periodic orbit which comes into existence through a saddle node. A jump in a pre-existing basic set is typically brought about through a tangency.

In analyzing tangencies of stable and unstable manifolds of a fixed point, issues such as the relative size of eigenvalues at the fixed point or the side on which the tangency occurs can determine whether their is an explosion or not. At certain tangencies, new basic sets are created and merge prior to the tangency; at others, they appear only after the tangency. We will illustrate examples of these phenomena in the sections that follow. In order to limit the possibilities, we restrict the analysis here to area-contracting, orientation-preserving planar maps. For such maps, fixed or periodic points without periodic points can suddenly appear or disappear in area-preserving maps, they do not occur when the map is area-contracting.

One of the purposes of this paper is to propose and discuss the following conjecture that says there are very few kinds of situations that can lead to explosions of generic families of maps. We consider a generic one-parameter family of diffeomorphisms \( F_\mu \) (or \( F \)) of the plane (or other two-dimensional space) that are area-contracting and orientation-preserving, i.e., \( 0 < \det DF_\mu(x) < 1 \) for all \( \mu \) and \( x \). We also assume there is some large compact set in phase space containing all \( y \) that are recurrent. For cases (3) and (4) below, we assume that for saddles \( p \) and \( q \) discussed in Section 3.2, \( DF_\mu(p) \) and \( DF_\mu(q) \) have positive eigenvalues. (If the eigenvalues are negative, we can usually consider \( F^2 \) instead.) For such a family, we conjecture that an explosion can only occur through four distinct mechanisms:

1. Isolated saddle-node bifurcations;
2. Embedded saddle-node bifurcations;
3. “Outer” homoclinic tangencies of a periodic orbit;

We emphasize that explosions in mechanisms (3) and (4) can only occur for rather special tangencies: the map must have an “outer tangency”. An “outer tangency” is a particular configuration of stable and unstable manifolds which we define in Section 3. We believe these are necessary conditions for explosions in generic systems. A related, less specific form of this conjecture was first given by Palis and Takens [44]. Giving detailed heuristic arguments for this conjecture would require a great deal of mathematical background. Suffice it to say, for area-contracting maps these mechanisms are the only mechanisms in which explosions have been seen.

Mechanism (2) involves an embedded saddle-node bifurcation, i.e., at \( \mu = \mu_c \) the saddle-node has recurrent points arbitrarily nearby; it is embedded in a larger set of recurrent points. If it is not embedded we call it isolated. Mechanism (1) of the conjecture above can be restated as follows. Generically, if a new recurrent set \( R \) appears and there are no recurrent points in some neighborhood of the new set, then we conjecture that \( R \) is a periodic orbit, and a saddle-node bifurcation has occurred. (Mechanism (1) can be illustrated by the bifurcation of a fixed point for \( x_{n+1} = \mu - x_n^2 \) at \( \mu = -0.25 \). No recurrent points exist for \( \mu \leq -0.25 \).) The primary aim of this paper is to explain mechanisms (3) and (4) of the conjecture regarding the tangencies.

In order to illustrate the conjecture, we describe seven numerical examples in Section 2. We present an embedded saddle-node bifurcation in a differential equation (Section 2.1), then an example of a chaotic attractor explosion (interior crisis) in Section 2.2. We also give an analogous example for a one-dimensional map, a combined example of an embedded saddle-node bifurcation and an interior crisis (Section 2.3). Next we present an interior crisis for a differential equation (Section 2.4), a basin boundary metamorphosis (Section 2.5), a chaotic saddle explosion [45] (Section 2.6), and a non-explosion case: a boundary crisis (Section 2.7).

In Section 3, we explain different types of tangencies. Then in Section 4 we use the description of tangencies to explain the numerical examples introduced in Section 2, including an explanation of the phenomenon of gap filling [46].
Remark. In the mathematics literature, there are different categories of recurrence. We stated the conjecture in terms of the recurrent set, however, it could be rephrased in terms of the "non-wandering" set or the "chain recurrent" set (see Section 3, [47–49] or [50] for definitions). Similar jumps in the non-wandering set are called $\Omega$-explosions (see, e.g., [51]). Palis and Takens [52] analyzed homoclinic tangencies and gave necessary conditions for such tangencies to be explosion points. (This article was also reproduced in [44].) Jumps in the chain recurrent set are called "chain explosions" in [53]. There it is shown that for a particular class of maps a necessary and sufficient condition for a heteroclinic tangency to be an explosion point is that it be on an outer cycle (defined in Section 3) and on no other type of cycle.

The conjecture here is based on the work cited above, together with many examples, some of which are presented and explained in this paper, in which the conjecture is true. Although chain components or basic sets in any of the recurrence categories are observed to have accessible periodic orbits at explosions, proving this fact (at least generically) remains an obstacle to showing the conjecture holds.

2. Numerical examples

2.1. Explosion caused by a saddle-node bifurcation

While varying a parameter of a system, a saddle-node bifurcation can lead to sudden changes in the dynamics of a system. Fig. 2 illustrates an embedded saddle-node bifurcation for the damped driven pendulum:

$$\ddot{x} + 0.2\dot{x} + \sin x = \mu \cos t.$$  (1)

Eq. (1) can be converted to a system of three coupled first-order ordinary differential equations. Furthermore, using a stroboscopic surface of section at times $t_n = 2n\pi$, we get a two-dimensional map. Fig. 2(a) illustrates two stable period 2 orbits (crosses). (These two attractors of (1) are symmetric partners of each other in that one is converted to the other under the transformation $x \rightarrow -x$, $t \rightarrow t + \pi$.) For the parameters of Fig. 2(a), both of these period 2 orbits have their largest eigenvalue (of the linearized map evaluated at the orbits) very close to (but smaller than) 1. Also shown in Fig. 2(a) is a non-attracting chaotic set (or chaotic saddle). We numerically find the chaotic saddle using the PIM triple procedure. In addition to the two period 2 attractors (nodes), there are also two period 2 saddles (not shown in the figure) whose unstable eigenvalues are close to (but larger than) 1. (There are also two fixed point saddles that are not shown.) For the parameter of the figure, one node is very close to one of the saddles, and the other node is very close to the other saddle. (The existence of two saddle-node pairs in this case is again a consequence of the symmetry of (1).) Increasing $\mu$ slightly, the two attracting period 2 orbits disappear, as they simultaneously merge with their complementary saddles, and a chaotic attractor appears (Fig. 2(b)).

The union of the basic sets in Fig. 2(a) (including two period 2 attractors, their complementary periodic saddles, and a chaotic saddle) is smaller than the chaotic attractor in Fig. 2(b). The gaps in the chaotic saddle are suddenly filled. Thus, there is an explosion as $\mu$ is decreased.

2.2. Explosion of a chaotic attractor (interior crisis)

A type of global bifurcation of a chaotic attractor frequently encountered in the literature is the interior crisis [1,2] (see [7–34] for experimental examples). As an illustration, we take the Ikeda map,

$$z_{n+1} = 0.85 + 0.9\exp \left(1 \left(0.4 - \frac{\mu}{1 - |z|^2}\right)\right),$$  (2)

which models a nonlinear optical cavity with feedback [56]. Here $z$ is a complex number, and, by taking real and imaginary parts, the map can be viewed as a two-dimensional real map. In Fig. 3(a), $\mu$ is chosen to

\footnote{A reference for terminology is [44]. Relevant references for $\Omega$-explosions are [44,51,52].}

\footnote{The acronym PIM in PIM triple procedure stands for "Proper Interior Maximum" [54,55].}
Fig. 2. A saddle-node bifurcation is an explosion. The forced damped pendulum: (a) For the parameter \( \mu = 2.1638 \), there are two period 2 attractors (crosses), two period 2 saddles (not shown), two fixed point saddles (not shown), and a non-attracting chaotic set (dots); (b) For \( \mu = 2.1639 \), the period 2 attractors are absent and a large chaotic attractor is created. This is an explosion.

be 7.2688 [57]. A chaotic attractor is shown in black, and a chaotic saddle in gray. Notice the gaps in the saddle. As \( \mu \) is increased past a critical value \( \mu_c \), the attractor suddenly jumps in size filling out the entire chaotic saddle (see Fig. 3(b)). The attractor now includes the old attractor plus the saddle and also fills in the gaps. The final set is therefore bigger than the union of the previous attractor and saddle. Thus, there is an explosion accompanying the merging of the attractor and the chaotic saddle. (This sudden transition takes place at the critical parameter value \( \mu_c = 7.26884894 [57] \).)

2.3. One-dimensional explosion in logistic map (interior crisis and embedded saddle-node)

Although, this paper is devoted to studying explosions for maps in the plane, such explosions are also observed for one-dimensional maps. We present an example in this section. In particular, consider the logistic map

\[
x_{n+1} = \mu x_n (1 - x_n)
\]

in the period 3 window. In Fig. 4 the attractor (in black), and the chaotic saddle (in this case a repelling

Fig. 3. An interior crisis explosion when a chaotic attractor and a chaotic saddle collide: (a) The Ikeda map for \( \mu = 7.2688 \). A chaotic attractor and a chaotic saddle are, respectively, drawn in black and gray. The gaps throughout the saddle remain large up to the critical parameter value \( \mu_c = 7.26884894 \); (b) Slightly larger parameter value for \( \mu = 7.2689 \). Only a bigger chaotic attractor remains; the chaotic saddle is now part of the attractor. The gaps of the previous figure are suddenly filled.
Fig. 4. A bifurcation diagram for the logistic map with the parameter \( \mu \) varying. Showing explosions from embedded saddle-node bifurcation at \( \mu_s \) and an interior crisis at \( \mu_c \). The region in black represents the attractor. The gray regions are chaotic repellors in a period 3 window. The widths of the white gaps within the repeller do not shrink to zero as \( \mu \) approaches \( \mu_s \) from above (\( \mu_s = 3.828 \ldots \)). At \( \mu_s \) there is a saddle-node bifurcation of two period 3 orbits; one is a regular saddle and one is an attracting orbit that becomes a flipped saddle (after period doubling). When the regular period 3 saddle touches the chaotic attractor at \( \mu_c \equiv 3.85680 \) (an interior crisis), all the gaps suddenly vanish, and the three-piece chaotic attractor enlarges instantaneously into a one-piece chaotic attractor.

Cantor set in gray) are plotted as functions of the parameter \( \mu \). Notice the white gaps throughout the saddle. Also drawn in black is the repelling period 3 orbit that collides with the three-piece attractor. This orbit originates from a saddle-node bifurcation at parameter value \( \mu_s \equiv 3.828 \ldots \) ("s" for saddle-node bifurcation). There is also another attracting period 3 orbit that goes through period doubling. At the parameter value corresponding to the collision of the three-piece chaotic attractor with the saddle created by the original saddle-node bifurcation (\( \mu_c \equiv 3.85680 \)), the white gaps suddenly become part of a larger one-piece chaotic attractor. Thus at \( \mu_c \), both an explosion (gap filling) and a merging of the chaotic and the three-piece attractor occur.\(^5\) If we now decrease the parameter value \( \mu \) from \( \mu_c \) to \( \mu_b \), a similar event occurs: a chaotic attractor suddenly appears and the gaps are filled at \( \mu_b \).

In Fig. 4, at \( \mu_s \) a period 3 attractor is created (increasing \( \mu \)) and has a basin of non-zero size (shown in white) including an interval and its pre-images. These intervals appear where there was previously an attractor (which was a large interval). Hence the saddle-node bifurcation at \( \mu = \mu_s \) destroys recurrent points, and there is an explosion as \( \mu \) decreases through \( \mu_s \).

2.4. Example of an interior crisis explosion for the forced Duffing equation

The forced doubled-well Duffing equation,

\[
\ddot{x} + \dot{x} - 10x + 100x^3 = \mu \sin(3.5t),
\]

exhibits a symmetric potential leading to interesting results. For \( \mu = 0.848 \), there are two disjoint chaotic attractors. Fig. 5 (a) shows one attractor, “A”, in its white basin of attraction. The other attractor, “B”, is drawn in a gray basin of attraction. (These two attractors are symmetric in that the transformation \( x \rightarrow -x, 3.5t \rightarrow 3.5t + \pi \) converts one attractor to the other.) There is also a non-attracting chaotic set labeled with thick black segments. Slightly increasing \( \mu \) to 0.850, the two attractors merge to form a bigger chaotic attractor (Fig. 5(b)). The gaps in the previous non-attracting set are suddenly filled. This is an explosion. This can be viewed as an interior crisis.

2.5. Jumps of basin boundaries which sometimes are an explosions (metamorphosis)

An example of the merging of two basic sets is the following. As \( \mu \) increases through the critical value \( \mu_c \equiv 1.315 \), the Hénon map,

\[
(x, y) \mapsto (\mu - x^2 - 0.3y, x),
\]

goes through a basin boundary metamorphosis [the boundary delimits the basin of attraction of a period 2 stable orbit (two medium size crosses in the white region) and \( \infty \) (gray region) in Fig. 6] (see [3]). Figs. 6(a) and (b), respectively, show the boundary before (\( \mu = 1.314 \)) and after (\( \mu = 1.316 \)) the metamorphosis. In Fig. 6(a) the boundary is smooth. There are two basic sets of interest. One is a fixed point saddle on the basin boundary (largest cross), the other is a chaotic saddle (small crosses). As explained in the discussion of inner tangencies in Section 3.1, just before

\(^5\)When an attractor suddenly jumps in size with variation of a parameter, it is usually called an interior crisis.
Fig. 5. Explosion of a chaotic attractor (interior crisis) for the forced double-well Duffing system: (a) The thin connected pieces $A$ and $B$ are two chaotic attractors ($\mu = 0.848$). The white and gray colors are for the basins of attraction of the two attractors. The thick black segments denote the non-attracting Cantor set; (b) For a slightly larger parameter ($\mu = 0.850$), a bigger attractor is suddenly created.

the metamorphosis, this chaotic saddle is very close to the boundary, and as the metamorphosis is approached it approaches the boundary.

Right after the metamorphosis (Fig. 6(b)), the formerly smooth (non-fractal) boundary suddenly becomes bigger and fractal (a “smooth–fractal basin boundary metamorphosis”). The chaotic saddle has merged with the fixed point basic set. This new bigger basic set is now embedded in the basin boundary. The basin boundary clearly goes through a sudden expansion. On the other hand, the new basic set is equal to the union of the chaotic saddle and the fixed point. No new recurrent points are created at the metamorphosis. In other words, there are no explosion points. Since no new parts are added to the resulting basic set, this bifurcation is not an explosion, i.e., the new set is entirely the result of two pre-existing basic sets merging.

Interestingly enough, a “fractal–fractal basin boundary metamorphosis” (the basin boundary is fractal before and after the bifurcation) is an explosion and can be viewed as two chaotic basic sets colliding. For the

Fig. 6. Smooth–fractal basin boundary metamorphosis: no explosion. For this figure and for Fig. 7, the basin of attraction of a period 2 attractor and of diverging solutions are, respectively, drawn in white and gray. The two medium size crosses in the white basin represent the attracting period 2 orbit. (a) For $\mu = 1.314$, the basin boundary is smooth. The two disjoint basic sets of interest are the chaotic saddle (small crosses) and an unstable fixed point (biggest cross on the basin boundary). Notice the smooth well-defined basin boundary. (b) We slightly increase the parameter $\mu$ and the basin boundary is fractal. Here the parameter value is $\mu = 1.316$. The chaotic saddle and the fixed point have merged, but no new recurrent points (explosion points) have been formed. On the other hand, the basin boundary has undergone a discontinuity (sudden expansion): it is now bigger and fractal. But the two basic sets (the chaotic saddle and the fixed point) have merged continuously. Also, the chaotic saddle is now on the basin boundary.
same Hénon map as before, but with $\mu \cong 1.395$, such a metamorphosis occurs. Before the event, the two basic sets on a collision course are non-attracting chaotic sets. One is on the basin boundary, the other in the basin of attraction. After the event, they merge and acquire explosion points. Details of such explosion are given in Section 4.

2.6. Explosion of a chaotic saddle

In this section we describe a sudden jump in the size of a chaotic saddle. We again use the Hénon map. For $\mu = 1.38031$, the map has two attractors; a period 2 periodic orbit and an attractor at infinity. Fig. 7(a) shows in gray the basin of attraction of $\infty$, i.e., the initial points of diverging solutions. The white region is the basin of attraction of a period 2 orbit represented by two large crosses. There is also a chaotic saddle, indicated by small crosses, in the white basin of attraction. In Fig. 7(b), $\mu$ is now increased by $10^{-5}$ to $\mu = 1.38032$. Notice that new points of the chaotic saddle (seen as small isolated crosses) have appeared at a non-zero distance from the previous set. These are explosion points. Thus, in this example, a basic set has gone through a sudden enlargement, gaining new points that were not near it or near any other basic set prior to the bifurcation. The mechanism responsible for this discontinuity is explained in Section 4. (This type of bifurcation was introduced in [45].)

2.7. A boundary crisis can be an explosion or not an explosion

For the Ikeda map $z_{n+1} = \mu + 0.9z\exp[i(0.4 - 6/(1 - |z|^2))]$ (using different parameters from the example of Section 2.2), a boundary crisis occurs at $\mu_c \cong 1.0024$ [58]. Such a crisis is characterized by the sudden disappearance of an attractor as the attractor collides with its basin boundary [1,2] as we vary a parameter. After the crisis, there is a recurrent set that approximates the old location of the attractor.

Fig. 8(a), just before a boundary crisis ($\mu = 1$), shows a chaotic attractor and fixed point (cross) with its stable manifold. This fixed point saddle along with its stable manifold are in fact the basin boundaries between two basins of attraction; one is for the chaotic attractor, the other for an attracting fixed point not shown in Fig. 8(a). The stable manifold is almost tangent to the chaotic attractor (which is on the closure of the unstable manifold of the fixed point saddle). Slightly past $\mu_c$, the chaotic attractor suddenly disappears, but a recurrent set (chaotic saddle) remains approximating the location of the former attractor. Fig. 8(b) shows the chaotic saddle for $\mu = 1.1$. The attracting set (chaotic saddle) is shown as a point (cross) with a stable manifold.
attractor) has changed nature into a non-attracting set (chaotic saddle).

This is not an explosion because the gaps seen in Fig. 8(b) are initially formed with zero widths at the crisis and the widths grow continuously from zero as we increase the parameter \( \mu \). Thus, new recurrent points at a finite distance from the basic set do not appear as \( \mu \) decreases through its crisis value. Fig. 8(c) shows the same saddle as in Fig. 8(b), but plotted with the fixed point saddle (cross) and part of its stable manifold. The portions of stable manifold in the form of tongues approach the chaotic attractor in Fig. 8(a), cross the chaotic saddle and continue to advance through the saddle (see Fig. 8(b)).

3. Tangencies

The stable set \( S(B) \) (respectively unstable set \( U(B) \)) of a set \( B \) is the set of points whose forward (respectively backward) orbits limit on \( B \). We call a periodic saddle \( p \) in a basic set \( B \) accessible (from the exterior of \( S(B) \)) if there is curve \( \gamma \) that intersects \( S(B) \) in exactly one point, namely \( p \). Often \( \gamma \) can be selected to be a branch of \( U(p) \). The stable set \( S(p) \) of a periodic point \( p \) has two branches which we denote by \( S(p, 1) \) and \( S(p, 2) \); similarly define \( U(p, 1) \) and \( U(p, 2) \) for the unstable set (see Fig. 9).

In all cases we have observed, if a pre-existing basic set with an accessible periodic orbit experiences an explosion, it must be by mechanism (2) and (3) or (4). In such situations, a universal scenario related to the tangency is always present. In order to describe this event, we describe a type of tangency called an outer tangency.

In addition to the concept of recurrent set defined in Section 1, we also use a weaker concept of recurrence called chain-recurrence [50]. An \( \epsilon \)-chain from \( x_0 \) to \( x_n \) is a sequence of points \( \{ x_0, x_1, \ldots, x_{n-1}, x_n \} \) such
Fig. 9. Labeling convention for heteroclinic tangencies: The fixed points are named \( p \) and \( q \) with the unstable manifold of \( p \) tangent to the stable manifold of \( q \). Each manifold is made of two branches: the branches \( S(p,1) \) and \( S(p,2) \) for the stable manifold of \( p \), and \( U(p,1) \) and \( U(p,2) \) for the unstable manifold of \( p \). The stable and unstable manifolds of \( q \) are named similarly. At the tangency point call the unstable branch \( U(p,1) \) and label the stable branch \( S(q,1) \). The branch of the unstable manifold of \( q \) on the same side of \( S(q,1) \) as the tangency is called \( U(q,1) \). Let \( S(p,1) \) be the branch of the stable manifold on the side of \( p \) towards which \( U(p,1) \) turns. Then all other branches are labeled with 2’s. If in addition \( U(q,2) \) crosses \( S(p,1) \) we call it inner. The tangency can be inner, outer or neither or both.

That \( |f(x_i) - x_{i+1}| < \epsilon \) for \( i = 1, \ldots, n-1 \). (It is also called an \( \epsilon \)-pseudo-orbit.) A point \( x_0 \) is called chain recurrent if, for each \( \epsilon > 0 \), there is an \( \epsilon \)-chain from \( x_0 \) to \( x_0 \) for some \( n \geq 1 \).

If explosion points are not recurrent then they are chain recurrent. Precisely at \( \mu = \mu_c \) the explosion point \( \tilde{y} \) is chain recurrent but, in general, will not be recurrent.

3.1. Homoclinic

A homoclinic inner tangency [59] (as shown in Fig. 10(a)) occurs when the branch of the unstable manifold that forms the tangency is tangent on the side of the stable manifold that it emanates from. Horseshoes are created before inner tangencies (\( \mu < \mu_c \)) and they contain an infinite number of different periodic orbits. In Fig. 10(b), stable and unstable manifolds of a periodic point are drawn for a situation before the occurrence of an inner tangency. Also shown in Fig. 10(b) is a box \( A \) which, when iterated several times, maps (e.g., \( F(A) = B \), \( F^2(A) = C \)) to \( C \), re-intersects \( A \) in a horseshoe shape. In this way, a horseshoe of the second iterate \( F^2 \) of the map is created, \( A \cap F^2(A) \) (see, e.g., [60,61]). In fact, as the tangency is approached, there is an infinite sequence of horseshoes created \( (A_m \cap F^m(A_m)) \) for increasing \( m \) that converge to the point of tangency. In this case, a point of tangency is not an explosion point because periodic saddles, formed before the tangency, exist in every neighborhood of the tangency point. From the definition in the introduction, an explosion point \( x \) has a distance \( \delta > 0 \) associated with it, so that for some arbitrarily small changes in the parameter of the map,
there are no recurrent points at or within $\delta$ of $x$. If, on the other hand, periodic saddles exist in every neighborhood of $x$, then $x$ is not an explosion point.

A homoclinic outer tangency [45] is illustrated in Fig. 11(a) and is characterized by the following: Let $U(p, 1)$ be the branch of the unstable manifold that is tangent to the stable manifold. Then $U(p, 1)$ emanates from the fixed point on one side of the stable manifold and approach the tangency from the other side of the stable manifold. If, as for the inner tangency (Fig. 10(a)), we consider the situation before the tangency, construct a rectangle $A$ crossing the unstable manifold (Fig. 11(b)), and iterate it, this time something different occurs. The box does not come back crossing the initial box. Instead, it is mapped to $F(A) = B, F(B) = C, F(C) = D, \ldots$. The resulting mapped box follows the other branch of the unstable manifold. Therefore no horseshoes are created before the tangency. As shown by Fig. 11(c), it is only after the tangency that it will be possible to create horseshoes. With a similar construction as before, we draw a box $A$ and iterate it ($F(A) = B, F(B) = C$). The iterates come back intersecting the initial box $A$. This intersection (e.g., $A \cap C \neq \emptyset$) implies the creation of new recurrent points (explosion points). Right after the tangency (see Fig. 11(c)), the tip of the unstable manifold barely goes through the stable manifold. Thus, horseshoes for $F^m$ only with $m$ very high can be initially formed, and consequently only very high periodic orbits will be present. Increasing the parameter further, lower period orbits will appear. In the simple outer tangency case shown in Fig. 11(a), the points of tangency are explosion points. Near the tangency, there are no recurrent points before the tangency ($\mu < \mu_c$). After tangency, horseshoes (and hence recurrent sets) appear.

Note that at $\mu = \mu_c$, the tangency points are chain recurrent. They are not recurrent because forward iterates of a tangency point $\tilde{x}$ converge to the fixed point $p$ never returning near $\tilde{x}$. On the other hand, to see that $\tilde{x}$ is chain recurrent, note that backward iterates also converge to $p$, $F^{-m}(\tilde{x}) \to p$ (along $U(p, 1)$). Thus, for any $\epsilon$, we can choose $n$ and $m$ large enough so that $|F^n(\tilde{x}) - F^{-m}(\tilde{x})| < \epsilon$. Hence, for arbitrarily small $\epsilon$, we can make an $\epsilon$ perturbation to the orbit from $\tilde{x}$ (namely, at time $n$ we perturb $F^n(x)$ by $F^{-m}(\tilde{x}) - F^n(\tilde{x})$), such that the perturbed orbit returns to $\tilde{x}$.

When the period of the periodic orbit is greater than one, a tangency can form a cyclic structure, and we call such a tangency a rotary tangency [59] (see Fig. 12). In such cases the branch of an unstable manifold of a periodic point $p_n$ becomes tangent with a branch of the stable manifold of a periodic point $p_{n'}$ from the same periodic orbit as $p_n$. All the tangencies shown occur simultaneously (i.e., at the same parameter value), resulting in the formation of closed circuit of invariant manifold segments (see next section). At rotary tangency, an inner (respectively outer) tangency is characterized by a branch of the unstable manifold of $p_n$ limiting on the same (respectively opposite) branch of the unstable manifold of the next point in the orbit $p_{n'}$.

![Fig. 11. An outer homoclinic tangency. A fixed point $p$ is represented with its stable and unstable manifolds.](image-url)

(a) The upper branch $U(p, 1)$ of the unstable manifold is tangent to the stable manifold from below at $\tilde{x}$ and therefore is an outer tangency. (b) Before outer tangency ($\mu < \mu_c$), the box $A$ is mapped to $F(A) = B, B$ to $F(B) = C$, and $C$ to $F(C) = D, \ldots$. No iterates come back intersecting the box $A$. This prevents the creation of horseshoes. (c) After outer tangency, a box $A$ can now be mapped on itself after some iterates, i.e., $A \cap C \neq \emptyset$. Horseshoes associated with an outer tangency can only be created after the outer tangency. (d) The definition of outer tangency is best defined in terms of “cycles” later in this paper, i.e., pieces of stable and unstable manifolds that form a closed path as shown here. The path lies above and below the stable manifold, so the cycle is an outer cycle and the tangency is an outer tangency.
Fig. 12. Rotary tangencies for a period 3 orbit, $p_1 \to p_2 \to p_3 \to p_1$: (a) Inner rotary tangency of a period 3 orbit; (b) Outer rotary tangency.

(We say a branch of the unstable manifold of $p_n$ and of $p_{n'} = F(p_n)$ are the same if the branch from $p_n$ is mapped to the branch from $p_{n'}$.)

Again, in this case, at an inner rotary tangency, horseshoes are created before the tangency, and at an outer rotary tangency, horseshoes are created after. These two types of rotary tangencies are used to explain the possible periods of orbits created at tangencies in [45, 59, 62].

3.2. Heteroclinic

Heteroclinic tangencies are tangencies between manifolds of different orbits. When two different orbits are involved in a tangency, we can still refer to inner and outer tangencies. To do this, we begin by defining "cycles".

A cycle is a closed path (a continuous image of a circle, not necessarily one-to-one) made of a finite number of oriented segments (following the dynamics) of stable and unstable manifolds of $p$ and $q$. Each segment has $p$ or $q$ as one of its end points. The stable and unstable segments must alternate. (Figs. 11(d), 13(b) and (d) and 17(b) show cycles.) Fig. 14 explains why we need not consider more than two fixed points.

A corner point of a cycle is a point where one unstable segment ends and a stable segment begins. A corner point can be a fixed point or periodic point, a point of tangency or a point of crossing of stable and unstable manifolds.

Fig. 9 shows the labeling convention used in the following. In the heteroclinic case, a cycle is an outer cycle if it contains the tangency point and segments of $S(p, 1)$ and $U(q, 2)$. In other words the cycle reverses sides. An outer tangency is a tangency point that is a corner point of an outer cycle (see Fig. 13(d)). See also Fig. 17 for the definition of a homoclinic outer cycle.

Fig. 13(a) shows a sketch of a heteroclinic inner tangency. The points $p$ and $q$ represent two fixed points. If $U(q, 1)$ intersects $S(p, 1)$, the points of tangency of $U(p, 1)$ and $S(q, 1)$ are chain recurrent. Similarly, in Fig. 13(a), points in the orbits of the heteroclinic points $x$ and $y$ are chain recurrent (but are not explosion points, because there are already arbitrarily nearby invariant sets that were created before the tangency). This can be understood with the following statement. The Smale horseshoe theorem

\footnote{Our analysis of heteroclinic tangencies, i.e., mechanism (4), does not use the assumption that $F_p$ is area decreasing. It is not even necessary at $p$ and $q$.}
Fig. 13. Inner and outer heteroclinic tangencies. Assume that the only branches that cross are those shown crossing. (a) Inner tangency of manifolds of two different orbits \( p \) and \( q \) (\( \mu = \mu_\perp \)). There are homoclinic crossings of the stable and unstable manifolds of \( p \) in every neighborhood of the heteroclinic points \( x \) and \( y \). The points \( x \) and \( y \) are chain recurrent but are not explosion points. (b) Inner cycle. (c) Outer tangency of manifolds of different orbits. The heteroclinic crossing point \( N_x \) is an explosion point. If \( U(q,1) \) does not cross \( S(p,1) \) (and thus, does not form an inner tangency), then the tangency point \( x \) is an explosion point. (d) Outer cycle.

implies that periodic orbits come arbitrarily close to each homoclinic point that is not a tangency point. These homoclinic points are present in Fig. 13(a) (crossings of the stable and unstable manifolds of \( p \)). Now consider Fig. 13(c) which shows an outer tangency of \( U(p,1) \) with \( S(q,1) \). There are two cases: (i) \( U(q,1) \) crosses \( S(p,1) \), and (ii) \( U(q,1) \) does not cross \( S(p,1) \). In case (i), the tangency is simultaneously an outer tangency (corresponding to the cycle shown in Fig. 13(d)) and an inner tangency (corresponding to the cycle shown in Fig. 13(b)). The point \( x \) of tangency of \( U(p,1) \) with \( S(q,1) \) is chain
Fig. 14. The conjecture asserts we must consider only tangencies involving one or two fixed points. What about the case of three fixed points? The tangency point $\tilde{x}$ is chain recurrent and there are three fixed points $p$, $q$, and $r$ in this figure. We need to consider heteroclinic cycles involving only two fixed points since a third fixed point $r$ which appears to be involved as in (a), can be eliminated by extending the manifolds of $p$ and $q$ as shown in (b).

We see that (Fig. 15(a)) a branch of the unstable manifold goes away from $p_3$, comes back toward it, crosses one branch of the stable manifold of $p_3$, to become tangent to the other branch of the stable manifold of $p_2$. This unstable manifold segment becomes tangent on the other side of the stable manifold of $p_2$. (The situation is as shown in Fig. 12(b).) It is therefore an outer tangency. Fig. 15(b) is an enlargement in the small box drawn in Fig. 15(a). It shows the manifold almost at tangency. Details of the scaling of periodic orbits created in an outer tangency are given in [45].

Now we explain the numerical example of Section 2.5 and Fig. 6 (the basin boundary metamorphosis). In this case, the mechanism of the bifurcation is an outer tangency involving $U(p, 1)$ crossing $S(q, 1)$. In case (ii), where $\tilde{x}$ is an outer tangency, but not an inner tangency, both $\tilde{x}$ and $\tilde{y}$ become explosion points as $U(p, 1)$ crosses $S(q, 1)$.

4. Geometry of the global bifurcations

This paper focuses primarily on the role of the outer tangency in explosions. We begin this section by explaining the numerical experiment of Section 2.6 (explosion of a chaotic saddle). Consider the chaotic saddle in Fig. 7(a). There is a dominant period 3 behavior in this set (as shown in Fig. 7(a)). In Fig. 7(b), after the explosion, this type of period 3 behavior is no longer present.

Fig. 15(a) is for the same parameter as in Fig. 7(b). We find only one period 3 regular saddle (one with positive eigenvalues). More precisely $DF^3(p_i)$ has positive eigenvalues where $p_i$ are the period 3 points. This orbit is designated by large crosses in Fig. 15(a) and is part of the chaotic saddle shown in Fig. 7. We then compute the stable and unstable manifolds of the regular period 3 orbit, and observe that they are about to form an outer rotary tangency for the value of $\mu$ in Fig. 7(a). (Only a small part of the manifolds are plotted so that we can more clearly identify the type of tangency.) Furthermore, the new parts of the chaotic saddle that are created after the explosion are in the immediate neighborhood of these tangencies. So the explosion of this basic set is due to a rotary homoclinic tangency from a period 3 mediating orbit.
inner homoclinic tangency. Fig. 16 (just before tangency, $\mu = 1.314$) shows the almost tangent branches of the stable and unstable manifolds of the fixed point $p$. This is an inner tangency. In this case points of tangency are not isolated (not at a finite distance) from a pre-existing basic set. Horseshoes are formed prior to the tangency (e.g., the intersection of $C$ and $A$ in Fig. 10(a)) and limit on the points of tangency. Therefore, the expansion of the chaotic basic set is only due to the merging with the fixed point $p$. Although there is a jump in the basin boundary (the stable set of the basic set containing $p$), it jumps to contain a pre-existing basic set.

It is important to note that certain examples of basin boundary metamorphoses can include explosion points. In particular, this happens when a basin boundary is already fractal and therefore contains a chaotic basic set. This chaotic set collides with another chaotic basic set leading to a merging of two chaotic basic sets with explosion points. (An example of this type of basin boundary metamorphosis occurs for the Hénon map for $\mu_c \cong 1.395$.)

The numerical example of Section 2.2 (the explosion of the Ikeda chaotic attractor) is a heteroclinic event triggered by a tangency of two mediating orbits of period 5. This Ikeda example requires a more detailed picture in order to explain the gap filling of the chaotic saddle (Fig. 3(a) and (b)). (The following explanation holds also for the Duffing equation of Section 2.4.) In the schematic Fig. 17, we illustrate this mechanism with one fixed point $p$ rather than a period 5 orbit (to ease the explanation). Let $U(p, 1)$ be the inner (or tangency) branch of the unstable manifold, and $U(p, 2)$, the outer branch of the unstable manifold. Also drawn are branches of the stable manifold of $p$, $S(p, 1)$ and $S(p, 2)$. The manifold $U(p, 1)$ is tangent to $S(p, 1)$. On the other hand, $U(p, 2)$ has already crossed both branches of $S(p)$. Thus an infinite number of layers of $U(p, 2)$ envelop (i.e., limit on) $U(p, 1)$ from both sides.

The gaps in the chaotic saddle before the tangency of $U(p, 1)$ and $S(p, 1)$ at $\mu = \mu_c$ can be understood
A homoclinic tangency that is inner and outer. The tangency point $x$ is not an explosion point but there are explosion points in $H$. Before and at tangency there is a trapping region. The area denoted by $R_1$ maps to $R_2$ and eventually maps to the attractor $A$. Segments like $I$ may map to $H$ in $R_1$ and so $I$ cannot have any recurrent points. Immediately after tangency the trapping region is destroyed or diminished and the segment $I$ can be recurrent, consisting entirely of new points. In this picture, $U(p, 1)$ has tangency with $S(p, 1)$; (b) The cycle is a homoclinic outer cycle because it lies on both sides (above and below) of the stable manifold of $p$. The right loop and the left loop are each homoclinic inner cycles, because they lie on one side of the stable manifold. Hence the tangency is both outer and inner.

with the following construction. Let $H$ be an interval on $U(p, 2)$ as indicated in Fig. 17. This interval is included in a region $R_1$ shaded gray in Fig. 17. The region $R_1$ maps to a region $R_2$ and orbits from points in $R_1$ (and $R_2$) eventually go to an attractor $A$. Therefore, points in $R_1$ are not recurrent. Hence, there are no recurrent points in any of the intervals of $U(p, 2)$ which eventually map into $R_1$ (e.g., the interval $I$ shown in Fig. 17). These intervals are in the gaps in Figs. 3(a) and 5(a).

For all $\mu > \mu_c$, we see (Fig. 3(b)) that the gaps disappear. To explain this sudden filling of the gaps, the numerical computation suggests that at $\mu = \mu_c$ the attractor $A$ is the closure of $U(p, 1)$, and in what follows we assume this to be the case. Consider the pre-image of $H$ labeled $I$ in Fig. 17. The interval $I$ will limit on $A$, and is thus not recurrent; $I$ is in a gap. After tangency, iterates of $I$ eventually cross $S(p, 1)$ and return to a neighborhood of $I$. Thus the gap immediately becomes filled after the explosion. This example is an inner and outer tangency (because of the inner and outer cycles).

Finally, we explain how the gaps are being formed in the boundary crisis example shown in Fig. 8. Recall that in Section 2.7 where we describe this crisis, we claimed that this event is not an explosion, and that, as $\mu$ increases through its crisis value $\mu_c$, the gaps (see Fig. 8(c)) in the chaotic saddle go continuously to zero. These gaps are caused by the stable manifold tongues. Notice that the tongues following the exact form of the sides of the gaps (see Fig. 8(c)). In order to understand this, consider a ball around the fixed point in Fig. 8(c). The ball is separated into two by the segment of the stable manifold emanating from the large cross (fixed point). Iterate the ball forward. The lower right-hand side of the ball is in the basin of attraction of a fixed point attractor. These points will not come back in a neighborhood of their initial condition. Thus they are not recurrent points, and no horseshoe can be created on this side of the stable manifold. On the other side, points map back in a neighborhood of their initial conditions. Horseshoe can be created on this side. Iterating the ball backward, it stretches along the stable manifold taking with it the part of the ball in the period 1 basin. Thus the gaps are on the side of the stable manifold leading to the basin of attraction of a fixed point attractor. This is not an explosion for two reasons: (i) the crisis is triggered by an inner tangency (there can be no explosion point at tangency), and (ii) the sets merging are a fixed point and a chaotic set (we would need at least two chaotic sets merging to form explosion points at heteroclinic crossings).
5. Conclusion

The aim of this work was to present a conjecture which puts different global bifurcations, such as crises (sudden changes in chaotic attractors [1,2]), metamorphoses (sudden changes in basin boundaries [3–6]), and explosions of chaotic saddles [45], into the same general setting. We have identified the following bifurcations as explosions: isolated saddle-node bifurcations, embedded saddle-node bifurcations (Section 2.1), interior crises (Sections 2.2 and 2.4), boundary crises when the boundary is fractal, fractal–fractal metamorphoses, and explosions of isolated chaotic saddles (Section 2.5). The two following bifurcations are not explosions: boundary crises with a smooth boundary (Section 2.7), and smooth–fractal metamorphoses (Section 2.5). An explosion can be caused by a tangency in two cases: (1) at tangency points for an outer tangency, and (2) at heteroclinic crossings of two (or more) chaotic basic sets when they collide (the crossings are at the explosion points). A collision of a chaotic basic set and of a non-chaotic basic set does not lead to an explosion.

Our main emphasis has been twofold: First, there are only a few kinds of explosions. Second, apparent discontinuous phenomena are not necessarily explosions (i.e., the basic sets may not change discontinuously). We saw that studying different types of tangencies involved in these events can lead to a general understanding of the mechanism behind discontinuous changes of solutions of dynamical systems. We observe that outer tangencies play a crucial role. Finally, we have shown that gap filling can also be understood on the basis of our considerations.

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