Fractal Basin Boundaries Generated by Basin Cells and the Geometry of Mixing Chaotic Flows

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Experiments and computations indicate that mixing in chaotic flows generates certain coherent spatial structures. If a two-dimensional basin has a basin cell (a trapping region whose boundary consists of pieces of the stable and unstable manifold of some periodic orbit) then the basin consists of a central body (the basin cell) and a finite number of channels attached to it and the basin boundary is fractal. We demonstrate an amazing property for certain global structures: A basin has a basin cell if and only if every diverging curve comes close to every basin boundary point of that basin.

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Mixing in chaotic flows generates certain coherent spatial structures. Experiments and computations have revealed that such structures are composed of many thin striations with an overall pattern that remains invariant over time ([1–6], and references therein). Shinbrot and Ottino [5] show that coherent structures are related to folds of horseshoe maps that are present in chaotic systems. They construct coherent structures by manipulating folds in some prototypical problems and they argue that the ability to construct such structures is of practical importance for the control of chaotic or turbulent extended systems such as fluids, plasmas, and coupled oscillator arrays. Alvarez et al. [4] analyze the evolution of fluid material lines by following the deformation of continuous closed filaments as they are stretched, advected, and folded by the time-periodic sine flow. Giona et al. [6] demonstrate that the geometry and topology of material lines in 2D time-periodic chaotic flows is controlled by a global geometric property that they refer to as asymptotic directionality. Based on their geometric approach they introduce the concept of a geometric global unstable manifold as an intrinsic property of a Poincaré map of the flow and address the relation between the geometric global unstable manifold and the global unstable manifolds of hyperbolic periodic points. They argue that asymptotic directionality provides evidence of global self-organizing structure which characterizes chaotic flows and that such flows are analogous to Anosov diffeomorphisms, which they believe are the basic prototypes of mixing systems. Furthermore, since material lines that are evolved by a chaotic flow are asymptotically attracted to the geometric global unstable manifold of the Poincaré map, they argue that the reconstruction of the global unstable manifold can be used to obtain a quantitative characterization of the topological and statistical properties of partially mixed structures. We will return to these results later.

The behavior of the physical system is determined by the global structures of the system such as strange attractors and boundaries of basins. This Letter shows that certain structures can characterize certain types of fractal basin boundaries [7]. When we introduced the notion of basin cell [8,9] (see also Fig. 1 and its caption), we found that when such cells exist, fractal basin boundaries can be characterized robustly. Robust structures [that is, those structures that persist under small (smooth) perturbations in the system] are particularly valuable in studying nonlinear dynamics where many structures (like chaotic attractors) can often be destroyed by arbitrarily small (smooth) perturbations. A trapping region is a region from which points cannot escape as time progresses. From now on, we say that a basin is the collection of initial conditions whose trajectories eventually enter the interior of some specified

FIG. 1. A basin cell for a two-dimensional diffeomorphism $F$ is a trapping region $T$ such that the boundary of $T$ is pieces of the stable and unstable manifolds of some period-$m$ orbit $P$. In this figure, the light grey region is a trapping region, and therefore it is a basin cell ($m = 3$). This basin cell has $2m$ sides, namely $m$ stable edges and $m$ unstable edges (pieces of the stable and unstable manifolds of the points of $P$). We also say that the orbit $P$ generates the basin cell. Note that at least one corner point is a homoclinic point. Each of the three dark grey regions is the initial part of a channel of the basin that has the light grey region as its basin cell. The corresponding basin has three channels and it has a fractal basin boundary since three of the six corner points of the basin cell are homoclinic points (a basin boundary is called fractal if it contains homoclinic points).
compact trapping region. In our approach to the theory of
basins, we examine pieces of the stable and unstable mani-
folds of certain saddle periodic points. A prototype of a
trapping region that is a basin cell is shown in Fig. 1. If
a basin has a basin cell as its trapping region, then we say
that the basin has a basin cell. Basin cells reveal a great
deal about the structure of the corresponding basin. For ex-
ample, the six-sided basin cell of Fig. 1 (the grey region)
was generated by a periodic orbit of period 3. The correspond-
ing basin can be viewed as the central body (basin cell)
plus three channels (dark grey) that connect to it. These
channels are infinitely long and wind in a very compli-
cated pattern without crossing each other. The channels
may vary greatly in thickness but generally get quite thin
as they wander back and forth [see the black and white re-
regions in Fig. 2(a)]. A special case of a part of the main
result of this Letter is the following. Start drawing an arbi-
trary curve \( C \) in the basin cell of Fig. 1; continue drawing
that curve \( C \) in any of the three channels; as the curve gets
arbitrarily long, follow the channel forever [see also the
black region in Fig. 2(a)]. We refer to such a curve \( C \) as a
diverging curve (or diverging path) \([10]\). Every diverging
curve has an amazing property: The curve will necessarily
come arbitrarily close to every point in the boundary of
the basin; that is, for every basin boundary point \( p \) and every
open neighborhood \( U \) of \( p \), the curve \( C \) will pass through
\( U \). Hence, we say that the limit set of the curve \( C \) is the
entire basin boundary. Figure 2(a) displays three basins of
which one has a basin cell that is topologically equivalent
to Fig. 1. Another basin in Fig. 2(a) has a basin cell gener-
at by a saddle fixed point and is shown in Fig. 2(b). The
corresponding fractal basin boundary [shown in Fig. 2(c)]
is invariant. On the other hand, there are many basins that
are not basins of basin cells. For example, if the trapping
region is a disk that is mapped into itself and if the cor-
responding basin has a smooth (that is, nonfractal) basin
boundary, then there is no homoclinic point and therefore
there is no basin cell. Our result describes precisely in
terms of diverging curves (like curve \( C \) above) when any
particular basin is the basin of some basin cell. The fractal
basin boundary of a basin that has a basin cell (generated
by a period-\( m \) orbit) is invariant, and (as we have shown
previously \([9]\) ) it equals the closure of the stable mani-
fold of the period-\( m \) saddle periodic orbit that generates
the basin cell.

The main purpose of this Letter is to report the following
result for two-dimensional systems: If a basin \( B \) satisfies
some mild hypotheses (which will be specified later), then
the basin \( B \) has a basin cell if and only if every diverg-
ing curve has the entire basin boundary of \( B \) as its limit
set. (The complete proof of this result will be published
elsewhere.)

More precisely, we consider a map \( F \) which is a two-
dimensional dissipative smooth diffeomorphism; that is,
both \( F \) and its inverse are differentiable and the partial
derivatives are continuous. Our result is applicable, for
example, to forced oscillators or to mixing chaotic flows
as discussed above. For two-dimensional systems, tradi-
tional approaches give no way to determine rigorously if
an attractor is chaotic or to determine if there is more than
one attractor. Hence, “basin” is ill defined in practice. In
order to avoid this problem, in \([9]\), we redefine a “basin”
as the set of points that enter a trapping region. We say, a
compact region \( Q \) is a trapping region if \( F(Q) \subset Q \) and
\( F(Q) \neq Q \). Hence, once a trajectory enters a trapping re-
gion, it cannot leave that trapping region, and there must
be at least one attractor located there. If \( Q \) is a trapping
region, then the basin of \( Q \) is the set of points which event-
ually map into the interior of \( Q \). A set \( B \) is a basin if
there exists a trapping region \( Q \) such that \( B \) is the basin of
\( Q \). The trapping regions we are interested in have piece-
wise smooth boundaries (that is, the boundaries consist

![Figure 2](https://example.com/figure2.png)

**FIG. 2.** There are three basins for the time-\( 2\pi/\omega \) (Poincaré) map of the forced oscillator \( \dot{x} + \gamma \dot{x} + x - x^2 = \mu \sin(\omega t) \) with parameter values \( \gamma = 0.1, \omega = 0.8, \) and \( \mu = 0.08 \). (a) The white and black area shows the basins of the fixed point attractors, and the grey area shows the basin of infinity (that is, the collection of initial conditions whose trajectories diverge). (b) The grey region shows the basin cell of the white basin of (a). The saddle fixed point \( P \) generates the basin cell. (c) This is the (fractal) basin boundary of each of the basins of (a). The boundaries of all three basins coincide (see the text). Therefore (applying the result discussed in text) every diverging curve through the channel of the white basin in (a) has the entire basin boundary as its limit set, and the same is true for the black basin, but here a curve can diverge along any of the three channels.
of finitely many, smooth curve segments). More explicitly, the trapping region is constructed so that its boundary consists of pieces of stable and unstable manifolds of a periodic saddle orbit that also lies on the boundary of the trapping region. This periodic orbit is said to generate this trapping region. The trapping regions constructed in this way are called basin cells [8,9]. The periodic saddle orbit may have to be chosen rather carefully to generate a basin cell. We point out that only a few of the infinitely many periodic orbits may generate a basin cell. In Fig. 2(a) there are three basins shown, two of which have basin cells that are generated by a period-3 orbit and a saddle fixed point, respectively. Figure 2(b) shows the basin cell of the white basin in Fig. 2(a) and its boundary consists of pieces of the stable and unstable manifolds of a saddle fixed point. The black basin in Fig. 2(a) has a basin cell which is topologically equivalent to Fig. 1. We point out that a basin cell determines both the structure of its basin and the global structure of the corresponding basin boundary. As in Fig. 1, if a basin has a basin cell, then the basin boundary is fractal. The result in this Letter implies that the limit set of any diverging path is the entire basin boundary. In this case, we also say that the diverging path is approaching the basin boundary and its limit set is the entire basin boundary.

We now specify the “mild hypotheses” required by the theorem. From now on, let $F$ be an orientation-preserving smooth diffeomorphism [that is, $\det DF(x) > 0$ for all $x$]. Let $B$ be a region in the plane such that (i) $B$ is not the whole plane (otherwise $B$ has no boundary) and (ii) $B$ is the basin of a connected, simply connected trapping region (has no holes). It follows that $B$ is an open, connected, simply connected region and $F(B) = B$. Let $P \subset \mathbb{R}^2$ be a periodic orbit of period $m$ such that (i) $P$ is saddle-hyperbolic (that is, for every point $p$ of $P$, the norm of one of the eigenvalues of $DF^m(p)$ is less than one and the other eigenvalue exceeds one); (ii) each point $p$ in $P$ is a “$B$-accessible point” (that is, there is a curve of finite length that starts in $B$ and $p$ is the first point of the boundary of $B$ that is hit by the curve); (iii) the stable manifold of $P$ has no tangency with the unstable manifold of $P$. Note that all $B$-accessible points are on the boundary of $B$. In addition, we assume that (i) the points of $P$ are the only $B$-accessible periodic points and (ii) infinity is a repellor for points in $B$. (If there are other $B$-accessible periodic points, then a result in [11] shows that every $B$-accessible periodic point is of period $m$, the period of $P$).

As an example, assume a particle of unit mass undergoes sinusoidally forced one dimensional motions in a potential $V(x) = x^2/2 - x^3/3$ with inertia and linear viscous damping ($\gamma > 0$),

$$\ddot{x} + \gamma \dot{x} + x - x^2 = \mu \sin(\omega t).$$  \hspace{1cm} (1)

To illustrate the result, we select $\gamma = 0.1$, $\omega = 0.8$, and $\mu = 0.08$. The time-$2\pi/\omega$ (Poincaré) map of the forced oscillator Eq. (1) has three basins, shown in Fig. 2(a): $B_1$ (white region), $B_2$ (black region), and $B_3$ (grey region). The basins $B_1$ and $B_2$ both have fixed point attractors. Basin $B_3$ is the collection of initial points whose trajectories diverge to infinity. We concentrate on the boundaries of basins $B_1$ and $B_2$. Basin $B_1$ is the basin of a basin cell $C_1$ [shown in Fig. 2(b)]. This basin cell is generated by a saddle fixed point $P_1$. Basin $B_2$ has a basin cell $C_2$ (generated by a period-3 saddle $P_2$) which is topologically equivalent to Fig. 1. In [9] we showed the following: Let $P$ be a periodic saddle orbit that generates a basin cell $C_P$ [12] and let $B_P$ be the basin of $C_P$. If the outer branch of the unstable manifold of $P$ intersects at least two other basins, then $B_P$ is a Wada basin and the boundary of $B_P$ is a Wada basin boundary (that is, every basin boundary point of $B_P$ is a boundary point of at least three basins). Kennedy and Yorke [13] reported a numerical example for the forced damped pendulum and argued that the Wada property (that is, every point on the boundary of one basin is on the boundary of all basins) should occur frequently in physical systems. Recently, Sweet et al. [14] reported the occurrence of the Wada property in a simple experiment in chaotic scattering. Returning to the basins of Fig. 2(a), the outer branch of the unstable manifold of $P_1$ [see Fig. 2(b)] intersects basins $B_2$ and $B_3$. Thus the boundary of $B_1$ is a Wada basin boundary (every point on the boundary of $B_1$ is also on the boundary of both $B_2$ and $B_3$), and $B_1$ is a Wada basin. Applying a previous result of the authors [9] yields the result that each of the three basins $B_1$, $B_2$, and $B_3$ is a Wada basin and that the boundaries of all three basins coincide, that is $\partial B_1 = \partial B_2 = \partial B_3$. Hence, the three basins have the property that each point on the boundary of any of the basins is also on the boundary of the other two. The corresponding fractal basin boundary [shown in Fig. 2(c)] is invariant, and it (as we have shown previously [9]) equals the closure of the stable manifold of the period-3 saddle periodic orbit $P_2$ and it also equals the closure of the stable manifold of the saddle fixed point $P_1$. Therefore, applying the result of this Letter, every diverging curve in the channel of $B_1$ or in any of the three channels of $B_2$ comes arbitrarily close to each of the basin boundary points.

We now consider some particular diverging curves. Let $B_P$ be the basin of a basin cell $C_P$ (where $P$ is a period-$m$ orbit). Select any point $a$ in the basin cell $C_P$ and iterate $a$ backwards in time and write $b = F^{-1}(a)$. Let $[a, b]$ be a curve segment that lies in the basin $B_P$ and that connects the points $a$ and $b$. The union $S$ of all backwards iterates of the curve segment $[a, b]$ from $a$ to $b$, $S = \cup_{n=0}^{\infty} F^{-n}([a, b])$, is a diverging curve and therefore $S$ approaches every point in the basin boundary $\partial B_P$ and it has $\partial B_P$ as its limit set. This particular choice of diverging curve provides an easy method for generating a basin boundary numerically. Previously, it has been shown [9] that the basin boundary $\partial B_P$ is equal to the closure of the stable manifold of the orbit $P$. We also can show that every saddle-hyperbolic periodic orbit $Q$ in the basin boundary $\partial B_P$ has the property that the closure of the stable manifold of $Q$ equals the basin boundary $\partial B_P$. We now return
to the geometric approach by Giona et al. [6]. The basin boundary $\partial B_P$ is the “geometric global unstable manifold” of the diffeomorphism $F^{-1}$, which is the closure of the unstable manifold of the periodic orbit $P$ for the diffeomorphism $F^{-1}$. In addition, for every saddle-hyperbolic periodic orbit $Q$ in the basin boundary $\partial B_P$, the closure of the unstable manifold of the orbit $Q$ for $F^{-1}$ equals $\partial B_P$.

To reiterate, basin cells generate fractal basin boundaries for the corresponding basins and do occur in a wide variety of physical circumstances and imply the following interesting properties: (1) The basin cell determines the global structure of the corresponding basin boundary which is the geometric global unstable manifold for $F^{-1}$. (2) Every diverging curve has the entire basin boundary as its limit set.

In this Letter, we have given rigorous, numerically verifiable conditions on when these phenomena occur.

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[7] It is important to keep in mind the definition of a boundary point. A point $x$ is a boundary point of a basin $B$ if every open neighborhood of $x$ has a nonempty intersection with basin $B$ and at least one other basin. The boundary of a basin is the set of all boundary points of that basin. For a basin $B$, we write $\partial B$ for the boundary of $B$. We say the boundary $\partial B$ is fractal if it contains a transversal homoclinic point. Examples with fractal basin boundaries are common and occur, for example, in the forced damped pendulum and the forced Duffing equation. Fractal basin boundaries have been studied extensively—see, for example, C. Mira, C.R. Acad. Sci. 288A, 591 (1979); S. Takesue and K. Kaneko, Prog. Theor. Phys. 71, 35 (1984); F.C. Moon and G.-X. Li, Phys. Rev. Lett. 55, 1439 (1985); S.W. McDonald, C. Grebogi, E. Ott, and J.A. Yorke, Physica (Amsterdam) 17D, 125 (1985); E.G. Swinn and R.M. Westervelt, Phys. Rev. A 33, 4143 (1986); C. Grebogi, E. Kostelich, E. Ott, and J.A. Yorke, Physica (Amsterdam) 25D, 347 (1987); J.M.T. Thompson, Physica (Amsterdam) 58D, 260 (1992).
[10] Let $\gamma: [0,1] \to B$. We say that $q \in B$ is a limit point of $\gamma(t)$ (as $t \to 1$), if for each neighborhood $U$ of $q$ and each $\epsilon > 0$, there exists $t$ such that $1 - \epsilon < t < 1$ and $\gamma(t) \in U$. We call the set of limit points of $\gamma(t)$ (as $t \to 1$), the limit set of $\gamma$. A definition of diverging curve requires the notions of Riemannian metric on the basin boundary and half open paths. For any differentiable path $\gamma: [0,1] \to B$, define the length $\ell(\gamma)$ by $\ell(\gamma) = \int_0^1 |\gamma'(s)| ds$. For every $p, q \in B$, define the “path distance” $d(p, q)$ between $p$ and $q$ by $d(p, q) = \inf_\gamma \ell(\gamma)$, where $\gamma$ ranges over all paths $\gamma: [0,1] \to B$ satisfying $\gamma(0) = p$, $\gamma(1) = q$. Two points $p, q \in B$ might be close in the usual sense but every path lying entirely in $B$ might be quite long, so $d(p, q)$ would be large. The distance $d(\cdot, \cdot)$ defined on $B$ is a metric and is called the Riemannian metric. We say that a diverging path is a diverging (in the Riemannian metric) half-open path $\gamma: [0,1] \to B$ such that $d(\gamma(0), \gamma(1)) \to \infty$ as $t \to 1$.
[12] For a periodic orbit $P$ of period $m$, let $R$ be the closed region enclosed by $2m$ arcs of the stable and unstable manifolds of $P$ such that $P$ is in the boundary of $R$ (see Fig. 1 for $m = 3$). We call the sides of $R$ stable and unstable edges. It is shown in [9] that if every unstable edge $E$ of $R$ is mapped into $R$, then $R$ is a trapping region. This condition is numerically verifiable.