Unstable dimension variability: A source of nonhyperbolicity in chaotic systems

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Abstract

The hyperbolicity or nonhyperbolicity of a chaotic set has profound implications for the dynamics on the set. A familiar mechanism causing nonhyperbolicity is the tangency of the stable and unstable manifolds at points on the chaotic set. Here we investigate a different mechanism that can lead to nonhyperbolicity in typical invertible (respectively noninvertible) maps of dimension 3 (respectively 2) and higher. In particular, we investigate a situation (first considered by Abraham and Smale in 1970 for different purposes) in which the dimension of the unstable (and stable) tangent spaces are not constant over the chaotic set; we call this unstable dimension variability. A simple two-dimensional map that displays behavior typical of this phenomenon is presented and analyzed.

Keywords: Hyperbolicity; Stable manifolds; Shadowing

1. Introduction

One of the basic ideas in the study of nonlinear dynamical systems is the concept of hyperbolicity. Suppose that \( f: \mathbb{R}^d \to \mathbb{R}^d \) is a continuously differentiable invertible map with an invariant set \( \Sigma \). We say that the set \( \Sigma \) is hyperbolic if the tangent space \( T_x \) associated with any point \( x \in \Sigma \) can be split into a direct sum decomposition \( T_x = E^u_x \oplus E^s_x \) such that:
- the splitting \( E^u_x \oplus E^s_x \) varies continuously with \( x \in \Sigma \) and is invariant insofar as \( Df(E^u_x, E^s_x) = E^u_{f(x)}, E^s_{f(x)} \);
- there exist constants \( K > 0 \) and \( 0 < \rho < 1 \) such that \( \| Df^n(x)y \| < K\rho^n \| y \| \) if \( y \in E^u_x \) and \( \| Df^{-n}(x)y \| < K\rho^n \| y \| \) if \( y \in E^s_x \).

Hyperbolic sets have convenient mathematical properties. If an invariant set \( \Sigma \) is hyperbolic, then (among other things) it can be shown that:
- stable and unstable manifolds can be defined for each \( x \in \Sigma \);
- \( \Sigma \) and its dynamics are structurally stable, that is, they are topologically unchanged if a small perturbation is added to \( f \);
- if a small amount of noise is added to a hyperbolic system with a chaotic attractor, then the noisy orbit is shadowed by a "true" orbit, that is, a given noisy...
orbit is closely followed for all time by an orbit of
the corresponding noiseless process.

The shadowing property, for example, is frequently
of great interest in practice. Often, one wants to know
whether there exists a “true” orbit that closely follows
a given computer-generated orbit (where the noise re-
results from roundoff error).

Most chaotic attractors that arise from maps of phys-
ical interest are not hyperbolic. In many of the cases
that have been studied to date, the attractors fail to be
hyperbolic because there exist points \( x \) within the at-
tractor whose stable and unstable manifolds intersect
tangentially (such intersections are called homoclinic
tangencies). At homoclinic tangencies, the subspaces
\( E_x^s \) and \( E_x^u \) are undefined.  

Abraham and Smale [2] gave an example of an in-
variant set \( \Sigma \) that can fail to be hyperbolic in a way that
is different from the often observed tangency mecha-
nism. In particular, they constructed a diffeomorphism
from \( T^2 \times S^1 \) to itself that has an invariant set for
which the dimension of \( E_x^u \) is either 1 or 2, depending
on \( x \). (In the remainder of this paper, we refer to the
dimension of \( E_x^u \) and \( E_x^s \) as the unstable dimension
and stable dimension, respectively, of the point \( x \)). We
call this situation unstable dimension variability. Be-
cause the unstable dimension is not the same at each
\( x \) on the invariant set, a continuous splitting \( E_x^u \oplus E_x^s \)
cannot exist, and thus there is no hyperbolicity. An-
other consequence of this is that typical trajectories on
\( \Sigma \) have arbitrarily long orbit segments for which, over
the orbit segment, the orbit on the average is repelling
in one dimension and also have arbitrarily long orbit
segments over which the orbit on average is repelling
in two dimensions.

Romeiras et al. [3], Dawson [4], and Dawson
et al. [5] observed similar behavior in a numeric-
ical study of a four-dimensional invertible map
describing a kicked double rotor. Apparently, the
attractor has points \( x \) at which the dimension of
\( E_x^u \) is 1 and points \( x \) at which the dimension of
\( E_x^u \) is 2. This situation can be inferred from the
numerical observation that there are fixed points
embedded in the attractor that have both two un-
stable directions and one unstable direction. As
further evidence, it is also observed that finite
time computations of the second Lyapunov ex-
ponent fluctuate between negative and positive values.
Kennedy and Yorke [6] have also discussed exam-
pies of dynamical systems with this heterogeneous
property.

Let \( n \) be a fixed positive integer, and let \( Df^n(x_0) \)
denote the Jacobian (first derivative matrix) of \( f^n \)
evaluated at \( x_0 \). The singular values of \( Df^n(x_0) \) are
\( \sigma_1(x_0, n) \geq \cdots \geq \sigma_d(x_0, n) \geq 0 \). We define the \( k \)th
time-\( n \) Lyapunov exponent associated with \( x_0 \) as

\[
\lambda_k(x_0, n) = \frac{1}{n} \ln \| Df^n(x_0)u_k \|, \tag{1}
\]

where \( u_k \) is the right singular vector of \( Df^n(x_0) \)
corresponding to \( \sigma_k(x_0, n) \).

Fig. 1 shows the distribution of \( \lambda_2(x_0, 50) \) for
the double rotor map for the case \( \rho = 8 \), discussed
in [5]. The curve was generated by computing the
time-50 Lyapunov exponents associated with \( 10^7 \) dif-
ferent initial conditions on the attractor and binning

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1 Despite the existence of homoclinic tangencies, it is possible
to prove the existence of shadowing orbits over long (but finite)
times for many computer-generated iterates of chaotic attractors
of two-dimensional invertible maps [1].
them into a histogram. Although the mean value of $\lambda_2(x_0, 50)$ is slightly less than 0, about half of the initial conditions examined show a net expansion in two directions and half in one direction. (Note that, although the finite-time exponent $\lambda_k(x_0, n)$ depends on $x_0$, the usual infinite-time Lyapunov exponent, $\lambda_k = \lim_{n \to \infty} \lambda_k(x_0, n)$, takes on the same, slightly negative value for almost every initial point $x_0$ with respect to the measure on the invariant set.)

This paper investigates in more detail how nonhyperbolic sets arise that have different numbers of expanding and contracting directions associated with the tangent space at each point. To do this, we introduce what we regard as the simplest prototypical model that exhibits this property. Our model is a noninertible two-dimensional map. For our model, we show that typical orbits on the invariant set come arbitrarily close to an infinite number of saddle periodic orbits (where the unstable dimension is 1 and the stable dimension is 1) and source periodic orbits (where the unstable dimension is 2 and the stable dimension is 0). We believe that the properties that we find for this map are to be expected in typical noninvertible maps of dimension 2 and greater, and in typical invertible maps of dimension 3 and greater. Note that, as is discussed in Section 2, the possibility of differing numbers of expanding and contracting directions on an ergodic invariant set is absent for invertible two-dimensional maps like the Hénon map.

2. The torus map

Our model is a map of the 2-torus $T^2$ given by

$$y_{n+1} = 2y_n \mod 2\pi,$$
$$z_{n+1} = (y_n + z_n) + A \sin(y_n + z_n) \mod 2\pi$$

(2)

where $A$ is a parameter such that $0 < A < 1$ and we write $x_n = (y_n, z_n)$. (In this paper, we focus on the case where $A = 0.6$.) The Jacobian matrix of the map is

$$Df(y_n, z_n) = \begin{pmatrix} 1 + A \cos(y_n + z_n) & 0 \\ 1 + A \cos(y_n + z_n) & 2 \end{pmatrix}.$$

There are two fixed (period-1) points of the map: $(0, 0)$ and $(0, \pi)$. The origin $(0, 0)$ is a source, because the eigenvalues of $Df(0, 0)$ are 2 and $1 + A > 1$. The point $(0, \pi)$ is a saddle, because the eigenvalues of $Df(0, \pi)$ are 2 and $0 < 1 - A < 1$. It is straightforward to verify that, for any initial condition, the largest Lyapunov exponent of the map is $\lambda_1 = \ln 2$ and that the second Lyapunov exponent satisfies $\ln(1 - A) \leq \lambda_2 \leq \ln(1 + A)$. (Here, and in what follows, the term Lyapunov exponent refers to the infinite-time Lyapunov exponent.)

Let $a_i = 1 + A \cos(y_i + z_i)$. Then, over any orbit $x_1, x_2, \ldots, x_n$, we have

$$Df(x_n)Df(x_{n-1}) \cdots Df(x_1) = \begin{pmatrix} 2^n & 0 \\ 2^n b_n & 2^n c_n \end{pmatrix},$$

(3)

where

$$c_n = \prod_{i=1}^{n} \frac{a_i}{2} \quad \text{and} \quad b_n = \sum_{j=1}^{n} c_j.$$

The eigenvectors of this matrix are $(0, 1)^T$, corresponding to the eigenvalue $c_n = \prod_{i=1}^{n} (\frac{1}{2} a_i)$, and $(1, d_n)^T$, where $d_n = b_n/(1 - c_n)$, corresponding to the eigenvalue $2^n$. The minimum angle $\theta$ between the two eigenvectors occurs when $d_n$ is largest, $\tan \theta = d_n^{-1} \max$. The largest possible value of $d_n$ is attained when $b_n$ and $c_n$ are largest ($c_n < 1$ since $\frac{1}{2} a_i < 1$). Since $a_i \leq 1 + A$, we can bound $d_n$ by setting $a_i = 1 + A$ for all $i$. (This bound is attained at the fixed point at (0, 0).) In this way, we obtain

$$\tan \theta \geq \frac{1 - A}{1 + A},$$

or, for the case $A = 0.6$ that we study numerically, $\tan \theta \geq 0.2$. Hence, the angle $\theta$ is bounded away from zero, and there can be no question of stable and unstable manifolds becoming tangent. Thus, the mechanism of nonhyperbolicity that occurs, for example, in the Hénon map is absent for Eq. (2).
The following theorem gives the basic properties of the map (2).

**Theorem 1.** Consider the map (2) with $A = 0.6$. Then

1. for almost every $x \in T^2$ (with respect to Lebesgue measure), the $\omega$-limit set of $x$ is $T^2$;
2. periodic saddle points are dense in $T^2$.

We also have the following conjecture.

**Conjecture 1.** For the map in Eq. (2) with $A = 0.6$:

1. for almost every $x \in T^2$ (with respect to Lebesgue measure), the forward time average of $x$ converges to a unique Sinai–Bowen–Ruelle measure $\mu$; that is,

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} \delta(f^k(x)) \to \mu;
$$

2. periodic sources are dense in $T^2$.

The conclusions of Theorem 1 are valid for $A$ sufficiently large; there is nothing special about the value $A = 0.6$. Appendix A discusses a theorem similar to Theorem 1 for a different map, whose proof can be adapted to prove Theorem 1.

Theorem 1 and the conjecture imply that the dynamics of the map (2) are qualitatively similar to those described by Abraham and Smale [2]. In particular, typical orbits come arbitrarily close to any number of saddles and sources in the torus. Thus, typical orbits have arbitrarily long sections where the map expands nearby points in one direction and arbitrarily long sections where the map expands in two directions.

The noninvertible two-dimensional map (2) can be extended to a three-dimensional invertible map with essentially the same properties. For example, we can append another dynamical variable $v_n$ that evolves according to

$$
v_{n+1} = \begin{cases} 
\lambda_a v_n & \text{for } 0 < y_n \leq \pi, \\
\lambda_b v_n + (1 - \lambda_b) & \text{for } \pi < y_n \leq 2\pi.
\end{cases}
$$

In this case, the equations for $v_{n+1}$ and $y_{n+1}$ yield the generalized baker’s map [7]. We expect that properties (1) and (2) in Theorem 1 occur in typical noninvertible maps of dimension 2 and higher, and in typical invertible maps of dimension 3 and higher. In fact, the proof Theorem 1 (see Appendix A) also implies the following corollary.

**Corollary 1.** The properties of the theorem for the map (2) continue to hold for sufficiently small $C^{1+\alpha}$ perturbations of Eq. (2).

Therefore, properties (1) and (2) of Theorem 1 are not expected to be exceptional and should occur in realistic examples, as indeed seems to be the case for the kicked double rotor map. Note, however, that the properties of Theorem 1 are absent for the chaotic attractor of an invertible two-dimensional map, such as the Hénon map. In particular, if the map is invertible, a source periodic orbit (where $\dim E^u_x = 2$ and $\dim E^s_x = 0$) cannot be embedded in the attractor. Chaotic orbits cannot approach source periodic orbits for invertible maps, because they repel in all directions. On the other hand, periodic orbits that attract in two directions ($\dim E^u_x = 2$ and $\dim E^s_x = 0$) are themselves attractors; therefore, they cannot exist as embedded points in a chaotic attractor. Thus, the only embedded periodic orbits of invertible two-dimensional maps are saddles ($\dim E^s_x = \dim E^u_x = 1$).

3. **The distribution of finite-time Lyapunov exponents**

Theorem 1 and Conjecture 1 imply that the second Lyapunov exponent of the map (2), computed over finite-time segments of an orbit, fluctuates from positive to negative values. (The largest Lyapunov exponent computed from the matrix in Eq. (3) is $\ln 2$.)

The distribution of the second finite-time Lyapunov exponent is of some interest. For example, it can be shown [5] that it quantifies the likelihood that a given initial point is shadowable for some number of iterates. Let $P(\lambda, n)$ denote the probability density function of $\lambda_2(x, n)$ when $x$ is chosen randomly according to the measure on the invariant set $\Sigma$. By definition, the quantity $P(\lambda, n) \, d\lambda$ is the probability that $\lambda_2(x, n)$ lies...
between $\lambda$ and $\lambda + d\lambda$. In other words,

$$\langle F(\lambda_2(x, n)) \rangle = \int F(\lambda) P(\lambda, n) \, d\lambda,$$

where the angle brackets denote the average of $x$ over the invariant measure $\mu$.

A numerical calculation of $P(\lambda, n)$ can be done as follows. Pick many randomly chosen initial conditions on the torus. Iterate each initial condition $x_0$ a large number of times (say 500). Then follow each orbit for a long time; after every $n$ steps, compute the second Lyapunov exponent from the second eigenvalue of the matrix in Eq. (3),

$$\lambda_2(x_0, n) = \frac{1}{n} \sum_{k=1}^{n} \ln(1 + A \cos(y_k + z_k)).$$

Fig. 2 shows a plot of the probability density of the time-10 Lyapunov exponent, $P(\lambda_2, 10)$, computed using the above method for $2 \times 10^7$ initial points. The map parameter is fixed at $A = 0.6$. The numerical computation suggests that about 20% of the initial conditions exhibit a net expansion in two directions over a sequence of 10 iterations.

The chaotic nature of the orbits implies that the numbers $\lambda_2(x, n)$ can be regarded as random variables. For $n$ sufficiently large, it can be argued [8] that $P(\lambda, n)$ has the form

$$P(\lambda, n) \sim \left( \frac{nG''(\lambda)}{2\pi} \right)^{1/2} e^{-nG(\lambda)},$$

where $\lambda$ is the infinite-time Lyapunov exponent (that is, $\lambda = \lim_{n \to \infty} \lambda_2(x_0, n)$ for almost every $x_0 \in \Sigma$). The minimum value of the function $G$ is 0 and occurs at $\lambda = \bar{\lambda}$, and $G(\bar{\lambda}) = G'(\bar{\lambda}) = 0$ and $G''(\bar{\lambda}) > 0$.

The utility of the form (4) is that it specifies $P$, which is a function of two variables $\lambda$ and $n$, in terms of a function $G$ of one variable $\lambda$. Eq. (4) suggests the following procedure. Define

$$\tilde{G}(\lambda, n) = -\frac{\ln P(\lambda, n) + \frac{1}{2} \ln n + C(n)}{n},$$

where the constant $C$ is of order 1 and is chosen so that the minimum value of $\tilde{G}$ is zero. The quantity $\tilde{G}$ can be obtained from a numerical computation of $P(\lambda, n)$ (see Fig. 2). The implication inherent in Eq. (4) is that $\lim_{n \to \infty} \tilde{G}(\lambda, n)$ is a function of $\lambda$ (namely, $G(\lambda)$) that is independent of $n$.

Fig. 3 shows the numerical estimates of $\tilde{G}(\lambda, n)$, derived using the above procedure, for $n = 10, 15, 20$, and 25. Thus, we see, as implied by Eq. (4), that for large $n$, the function $\tilde{G}(\lambda, n)$ as computed from Eq. (5) apparently asymptotes to a limit that is independent of $n$.

One difficulty is that we have poor statistics near $\lambda_- = \ln(1 - A)$ and $\lambda_+ = \ln(1 + A)$, because typical
orbits spend relatively little time near the saddle at 
(0, π) and the source at (0, 0), where the local ex-
pansion factors are close to the values λ− and λ+, re-
respectively. (For the value A = 0.6 considered here,
λ− ≈ −0.916 and λ+ ≈ 0.470.)

Nevertheless, it is possible to estimate how \(\tilde{G}(λ, n)\)
scales near these extreme values. Consider for instance
the saddle fixed point at (0, yr). Let \(x_0 = (y_0, z_0)\) be
an initial condition in a small box around \((0, π)\). The
horizontal coordinate expands by a factor of 2 on each
iteration, and the vertical coordinate is compressed by
a factor that is approximately \((1 + A \cos(πk + π))\) on
the kth iteration. Therefore,

\[
\lambda_2(x_0, n) ≈ \frac{2^{n-1}}{n} \sum_{k=0}^{n-1} \ln(1 - A \cos y_k).
\]

We expand the above expression about \((0, π)\) and use
the fact that \(y_k = 2^k y_0\) to obtain

\[
\lambda_2(x_0, n) ≈ \ln(1 - A) + \left( \frac{A}{1 - A} \right) \left( \frac{y_0^2}{6n} \right) (4^n - 1).
\]

Therefore, near \(\lambda_2 = \lambda_- = \ln(1 - A)\), we have
\(P(λ, n) = P(λ(x_0), n)\). The natural measure of the
attractor is smooth along the expanding direction,
which at \((0, π)\) has a nonzero y component. Thus,
\(d y_0 \approx P(λ, n) dλ\) by the definition of \(P\), so \(P(λ, n) \approx\)
d\(y_0 / dλ\). This implies

\[
P(λ, n) ≈ \frac{1}{2\sqrt{\lambda - \lambda_-}} \sqrt{\frac{6n}{4^n} \left( \frac{1}{A - 1} \right)}
\]

for \(λ\) near \(\lambda_-\). We substitute this expression into
Eq. (5) to obtain

\[
\tilde{G}(λ, n) ≈ \frac{1}{2n} \ln(\lambda - \lambda_-) + \ln 2
\]

for \(λ\) near \(\lambda_-\). Analogous results apply near the source
at \((0, 0)\), where we find

\[
\tilde{G}(λ, n) ≈ \frac{1}{2n} \ln(\lambda_+ - \lambda) + \ln 2
\]

for \(λ\) near \(\lambda_+ = \ln(1 + A)\).

Eqs. (6) and (7) imply that \(\tilde{G}\) asymptotes to \(\ln 2\) for
either \(λ\) near \(\lambda_-\) and \(λ > \lambda_-\) or for \(λ\) near \(\lambda_+\) and
\(λ < \lambda_+\).

The dashed straight line segment in Fig. 3 goes
through the point \((λ, G) = (λ_-, \ln 2)\) and is tangent to
the asymptotic computed \(G\). We believe that as \(n \to \infty\),
the computed \(G\) approaches this dashed line. (This
conjecture is based on the previous analysis in [9] of
related situations.) However, the behavior of \(\tilde{G}\) for
large \(n\) and for \(λ\) near \(\lambda_-\) depends on how these limits
are approached. For a fixed \(n\), the value of \(\tilde{G}\) tends
to infinity as \(λ \to \lambda_-\) (and likewise for \(λ \to \lambda_+\)). Good
numerical estimates of \(\tilde{G}\) are difficult to obtain for \(λ\)
near \(\lambda_-\); from Eq. (7), the required number of iterates
is \(n \gg 2 \ln 2 / \ln(\lambda_+ - \lambda)\). For values of \(λ\) away
from \(\lambda_-\) and \(\lambda_+\), we expect numerical estimates of
\(\tilde{G}\) to approach the dashed line as indicated in Fig. 3.
Our numerical results appear to be consistent with this
expectation.

4. Conclusion

Nonhyperbolicity in the most intensively studied
class of dynamical systems, namely, smooth two-
dimensional invertible maps, arises exclusively as a
result of tangency between stable and unstable man-
ifolds. In this paper, however, we emphasize that as
soon as the system becomes slightly more complex
(e.g., either a two-dimensional noninvertible map or a
three-dimensional invertible map), a completely dif-
f erent mechanism for nonhyperbolicity arises. We call
this mechanism unstable dimension variability. In this
paper, we have introduced a very simple model sys-
tem in which this mechanism is conveniently studied.
We present rigorous analysis and computer experi-
ments to illustrate the essential behaviors of systems
with this property. We anticipate that our model will
also prove useful for other studies of systems with
unstable dimension variability.

Appendix A. Outline of the proof of Theorem 1

The proof of Theorem 1 is quite complicated, and
the details provide little insight into the dynamics of
the map. In this appendix, we prove an analogous theorem for a different family of maps, whose proof retains the essential concepts needed to prove Theorem 1 but is simpler to present.

We consider the following family of 3-to-1 maps of the torus

\[ F(y, z) = \begin{pmatrix} f_1(y, z) \\ f_2(y, z) \end{pmatrix} \]

\[ = \begin{pmatrix} 3y \\ z - \delta \sin z + \epsilon(1 - \cos y) \end{pmatrix}, \]  

(A.1)

where each coordinate in the torus \( T^2 \) is taken modulo \( 2\pi \). The parameters \( \epsilon \) and \( \delta \) are chosen so that \( 0 < \epsilon < 0.1 \) and \( 0 < 20\delta < \epsilon \). With these restrictions, the following theorem applies.

**Theorem A.1.** Let \( F \) be given as above. Then

1. the orbit starting from almost every \((y, z) \in T^2\) (with respect to Lebesgue measure) has an \( \omega \)-limit set that is dense in \( T^2 \);
2. periodic saddle points are dense in \( T^2 \).

As with Theorem 1, we also conjecture:

**Conjecture A.1.** For almost every \((y, z) \in T^2\) (with respect to Lebesgue measure), the forward time average converges to a unique Sinai–Bowen–Ruelle measure \( \mu \); that is,

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} \delta(F^k(y, z)) \to \mu. \]

Before we discuss the proof of Theorem A.1, we first state and prove the following lemma.

**Lemma A.1.** Let \( F \) be given as above. The unstable manifold of the fixed point \( O = (0, 0) \) is dense in the torus \( T^2 \).

**Proof.** Let \( R \) be the region depicted in Fig. 4(a). This region is a band of vertical height \( 5\epsilon \) that is bounded below by a portion of the unstable manifold of \( O \) that contains \( O \) and extends for a length \( \pi \) in both the positive and negative \( y \) direction. The image of the upper boundary of \( R \) lies above the upper boundary of \( R \); therefore \( F(R) \) contains \( R \). In addition, the image under \( F \) of the bottom boundary of \( R \) is contained in \( R \). (Fig. 4(b)) shows \( F(R) \). Finally, \( \partial f_2 / \partial z \) at points in \( R \) is close to \( 1 - \delta \); in fact, it can be shown that \( 1 - \frac{3}{2}\delta \geq \partial f_2 / \partial z \leq 1 - \frac{1}{2}\delta \).

Let \((y_0, z_0) \) be an arbitrary point in \( R \). Since \( F(R) \supset R \), there exists at least one preimage of \((y_0, z_0) \) in \( R \);
let \((y_1, z_1)\) be one such preimage. Since \((y_1, z_1) \in R\), it also has a preimage, say \((y_2, z_2)\), that is in \(R\). In this way, we can produce an infinite sequence \(\{(y_k, z_k)\}\) of points in \(R\) with the property that \(F(y_k, z_k) = (y_{k-1}, z_{k-1})\).

Consider the point \((y_n, z_n)\) and the vertical line segment \(y_n\) that extends from \((y_n, z_n)\) to the unstable manifold of \(O\) that forms the lower boundary of \(R\). Let \(y_{n-k}\) denote the \(k\)th iterate of \(y_n\). By the construction of \(R\), the length of \(y_n\) is at most \(5\epsilon\). It is easy to see that \(y_j\) is contained in \(R\) for each \(0 \leq j \leq n\). Since \(|\partial f_2/\partial z| < 1 - \frac{1}{2}\delta\) for any point in \(R\), we see that the length of \(y_{n-k-1}\) is at most \((1 - \frac{1}{2}\delta)^n\).

The top point of \(y_0\) is \((y_0, z_0)\) and the bottom point of \(y_0\) is a point on the unstable manifold of \(O\). Since \(n\) is arbitrary and \(\delta > 0\), we see that the unstable manifold of \(O\) is arbitrarily close to \((y_0, z_0)\). Because \((y_0, z_0)\) is an arbitrary point of \(R\), the unstable manifold of \(O\) is dense in \(R\).

We now show that the iterates of \(R\) cover \(T^2\). Consider the dynamics of \(F\) on the circle \(C\) defined by \(y = z\). It can be shown that the iterates of any interval of length \(5\epsilon\) in \(C\) cover \(C\). In this way, the iterates of \(R\) cover a neighborhood of \(C\), and thus, since \(F\) expands by a factor of 3 in the horizontal direction, the iterates of \(R\) cover \(T^2\). \(\Box\)

Let \(A\) be a Lebesgue measurable set. We say that a point \(x \in A\) is a point of Lebesgue density for \(A\) if

\[
\lim_{r \to 0} \frac{m(A \cap B_r(x))}{m(B_r(x))} = 1,
\]

where \(m\) denotes Lebesgue measure and \(B_r(x)\) denotes a neighborhood of size \(r\) around \(x\). Recall that if \(A\) is a set of positive Lebesgue measure, then almost every point of \(A\) (with respect to Lebesgue measure) is a point of Lebesgue density for \(A\). Without loss of generality, we use square neighborhoods whose sides are parallel to the \(y\) and \(z\) axes in the proof of Theorem A.1.

**Proof of Theorem A.1** Let \(A\) denote the set of points whose forward orbit does not intersect a \(\beta > 0\) neighborhood of some point \(x = (y, z)\). We assume that the Lebesgue measure of \(A\) is positive. If no such \(x, \beta,\) and \(A\) exist, then we are done with part (1).

Otherwise, we suppose that such an \(x, \beta,\) and \(A\) exist. We will show that there exists a positive number \(\eta\) such that for any horizontal line segment \(l\), the fraction of points in \(l\) (with respect to one-dimensional Lebesgue measure) whose forward orbit intersects the \(\beta\) neighborhood of \(x\) is at least \(\eta\). This implies that \(A\) cannot have any points of one-dimensional horizontal Lebesgue density. By Fubini's theorem, \(A\) cannot have any points of two-dimensional Lebesgue density, and therefore \(A\) cannot have positive two-dimensional Lebesgue measure.

By Lemma A.1, the unstable manifold of \(O\) is dense in \(T^2\). Therefore, there exists a small piece of the unstable manifold of \(O\) that intersects the square \(\beta\) neighborhood of \(x\) near the middle. (The slope of any small section of the unstable manifold of \(O\) can be shown to be less than \(\frac{1}{2}\).) Label this piece of the unstable manifold \(\xi\).

Let \(l\) be an arbitrary horizontal line segment. Without loss of generality, suppose that the length of \(l\) is \(2\pi/3^n\) for some integer \(n\). Let \(l_k\) denote the \(k\)th iterate of \(l\). Clearly, the horizontal length of \(l_k\) is \(2\pi 3^k/3^n\), and \(l_{n+8}\) crosses the circle \(y = 0\) several times. Some of these crossings may be very close to the source at \((0, \pi)\); however, at least one of them is a distance \(\epsilon\) from \((0, \pi)\) and continues for a horizontal length of \(\frac{1}{2}\) to both sides of the circle \(y = 0\). \(^3\)

Let \(\psi_k\) denote a piece of \(l_{n+8}\), and let \(\psi_k\) denote the \(k\)th iterate of \(\psi\).

The well-known lambda lemma implies that if \(k\) is sufficiently large, then \(\psi_k\) approaches any fixed length of the unstable manifold of \(O\) to any desired degree of accuracy. In fact, for any fixed length (in this case, long enough to include \(\xi\)), and any fixed

\[^3\text{Notice that the magnitude of the slope of the unstable manifold is less than }\epsilon,\text{ because the set }\{(u, v) : |v| \leq \epsilon|u|\}\text{ is a cone that is mapped into itself in the tangent space. Choose a point on } l_n\text{ that has the form }x_n = (\pi, z_n)\text{. Let }x_{n+k} = (\pi, z_{n+k}) = F^k(x_n)\text{. We claim that for some }k \leq 8,\text{ we have }|z_{n+k}| \geq 6\epsilon,\text{ which also implies that at least one of the crossings of }l_{n+8}\text{ and the circle }y = 0\text{ satisfies the stated property. Notice that }z_{n+1} = z_n - \delta \sin z_n + 2\epsilon \geq z_n + \frac{1}{2}\epsilon.\text{ If }|z_n| < 6\epsilon,\text{ then }z_{n+8} \geq z_n + 8(\frac{3}{2}\epsilon) > 12\epsilon - 6\epsilon = 6\epsilon.\]
degree of accuracy (in this case, uniform approximation to within $\frac{1}{4}\beta$), there is a number $N$ such that any approximately horizontal curve (i.e., slope less than $\frac{1}{2}$) that crosses the circle $y = 0$ at a horizontal distance of $\frac{1}{2}$ on either side and whose crossing is at least distance $\epsilon$ away from $(0, \pi)$ also approximates the unstable manifold of $O$ to this desired degree of accuracy.

The horizontal measure of $\psi_N$ is $3^N$, and the horizontal measure of $\psi_N \cap B_\beta(x)$ is at least the horizontal measure of $\zeta$, which is $\beta$. Moreover, the horizontal measure of $l_{n+N+3}$ is $2\pi 3^{N+3}$ and the horizontal measure of $l_{n+N+3} \cap B_\beta(x)$ is at least $\beta$. Therefore, the fraction of points (with respect to one-dimensional Lebesgue measure) in $l$ whose forward orbits intersect $B_\beta(x)$ is at least $\beta/(2\pi 3^{N+3})$. This suffices for our choice of $\eta$ and concludes the proof of part (1).

To prove part (2), we consider a square region that we suppose contains no periodic saddle, then show that some iterates of this square is mapped through itself. Let $B$ be a square neighborhood of size $\eta = 2\pi/3^n$, centered about a point $x$, whose sides are parallel to the axes. Let $l$ be the top boundary of $B$. The segment $l$ behaves in the same way as the segment $l$ discussed in the proof of part (1). By choosing $n$ larger than

$$\log(\epsilon/\pi) / \log((1 + \delta)/3),$$

we guarantee that the bottom of $F^k(B)$ remains within a vertical distance $\epsilon/4$ of the top for $1 \leq k \leq n + 3$, so $F^{n+3}(B)$ crosses the circle $y = 0$ at least once at a point that is at least $\frac{1}{4}\epsilon$ away from the source at $(0, \pi)$. Therefore, by choosing $N$ in a manner analogous to the proof of part (1), we can show that $B_{n+N+3}$ crosses $B$ in the required way. □

We conclude this section with a proof of the following result, which provides some support for the conjecture described in Section 2.

**Theorem A.2.** Let $F$ be given by (A.1). For a set of positive Lebesgue measure in the $(\epsilon, \delta)$ parameter space, periodic sources are dense in $T^2$.

**Proof.** As in the proof of part (2) of Theorem A.1, we consider a square region that we suppose contains no periodic source, then show that some iterate of this square is mapped through itself. Consider the dynamics of $F$ on the circle $y = \pi$. From standard circle map theory, we know that there is a set of positive measure in $(\epsilon, \delta)$ parameter space such that the map on the circle is conjugate to an irrational rotation of the circle. Let $\epsilon$ and $\delta$ be chosen so that the map $F$ restricted to the circle $y = \pi$ is conjugate to an irrational rotation.

As usual, we consider a square neighborhood $B$, centered about $x$, whose sides are parallel to the axes and have length $2\pi/3^n$. Clearly, $F^{n+3}(B)$ crosses the circle $y = \pi$ in several places and contains a neighborhood (not necessarily square) of an interval $\gamma$ on the circle $y = \pi$. Since the map $F$ restricted to the circle $y = \pi$ is conjugate to an irrational rotation, a finite collection of iterates of $\gamma$ can cover the circle $y = \pi$. Therefore, a finite collection of iterates of $B$ covers a thin annular strip containing $y = \pi$ of horizontal width $2\pi/3^k$ for some $k$. It is straightforward to see that the preceding collection of iterates of $B$ cover all of $T^2$ when they are iterated another $m$ times. In particular, some iterate of $B$ contains the source $(0, \pi)$. □

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**References**

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