The topology of stirred fluids

Judy A. Kennedy a,*, James A. Yorke b

a Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA
b Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA

Received 11 November 1994; revised 16 March 1996, 4 October 1996

Abstract

There are simple idealized mathematical models representing the stirring of fluids. The models we consider involve two fluids entering a chamber, with the overflow leaving it. The stirring creates a Cantor-like, but connected, boundary between the fluids that is best described point-set topologically. We prove that in many cases the boundary between the fluids is an indecomposable continuum. © 1997 Elsevier Science B.V.

Keywords: Indecomposable boundary; Indecomposable continuum; Mild turbulence; Stirred fluids; Invariant open sets; Dynamical system; Discrete models; Sets of full measure; Dense residual sets

AMS classification: 54F20; 54F50; 58F12

1. Introduction

Our goal in this work is to study the boundary between two mixed fluids from a new perspective. Turbulence, however mild, is a chaotic phenomenon, which although not in general precisely defined, has been extensively studied. We investigate a highly simplified model of the flow of two viscous fluids (of equal density) in the plane. The fluid is stirred and has been stirred for all time in the past, and we examine the boundary at some instant in time. In many situations in a bounded container, the boundary would become dense and the fluids totally mixed. We however envision a container with fluid entering and leaving. No diffusion is allowed so the boundary remains sharp. We consider both the periodic case where the fluid is periodically squeezed and rotated and the aperiodic case.

* This research was supported in part by an NSF (Division of Mathematics) grant and a US Department of Energy (Office of Energy Research, Office of Scientific Computing) grant.
* Corresponding author. E-mail: jkennedy@brahms.udel.edu.
where the sequence of applications of $R$ (the squeeze map) and $J$ (the stirring or rotation map) is chosen at random. For each point $(x, y)$ at time 0, we can examine the trajectory as time approaches $-\infty$. If the trajectory diverges to $y = +\infty$ as time approaches $-\infty$, we say that $(x, y)$ is a point of the red fluid, while if the trajectory diverges to $-\infty$, we say that $(x, y)$ is a point of the blue fluid. Suppose that $A^*$ denotes the collection of all points in the plane that are neither red nor blue. (More precise statements follow.) Let $\tilde{A}$ be the 2-point compactification of $A^*$ obtained by adding the points $\infty^+_x = (\infty, 0)$ and $\infty^-_x = (-\infty, 0)$. Then for our model, $\tilde{A}$ is a continuum. We give examples in the periodic case where $\tilde{A}$ has no interior, is indecomposable, and is homeomorphic to a Knaster continuum. (See Theorem 13.) The periodic case is the study of a map $F$ (a composition of $R$'s and $J$'s that repeat periodically). Hence, it is a dynamical system and $A^*$ is invariant under $F$. Many papers study continua that are invariant under maps.

The aperiodic case is somewhat different, but represents a clear physical situation. In the aperiodic case $\tilde{A}$ is generically (for generic sequences of $R$ and $J$) arclike and indecomposable (Theorem 26). A measure can be placed on the set of sequences in a standard manner, in which case we prove that for almost every sequence, $\tilde{A}$ contains (nontrivially) an indecomposable continuum that contains the points $\infty^+_x = (\infty, 0)$ and $\infty^-_x = (-\infty, 0)$, and $\tilde{A}$ has no interior (Theorem 27).

The usual approaches to fluid flow have been through partial differential equations such as the Navier–Stokes equations or through statistical mechanics. Here we consider a discrete time process consisting of two discrete maps from the plane to itself which model fluid flowing into a chamber, allowed to leave at the same rate at which it is entering, and stirred inside the chamber. The fluid is sufficiently viscous that it has the consistency of glycerin or even toothpaste. Since we wish to study the interface between the two fluids, called red and blue, entering the chamber, the fluid enters the chamber from two openings. (See Fig. 1.) The main object is to describe topologically the nature of the fluid boundary when the experiment has been going since time $t = -\infty$. At time $t = 0$, the observer, a topologist, looks inside the chamber and describes the fluid boundary. The nature of the interface of the fluids is an indication of the complexity of the behavior. In the future we hope to study the more complicated 3-dimensional case.

![Fig. 1. Schematic diagram of the experiment.](image-url)
In Figs. 2–9, computer generated pictures illustrate the results of the experiment in a number of different situations. The pictures were made with the software *Dynamics* [5]. Each considers a map of the form \( F = J \circ R \). Representative parameter values of \( \lambda \) and \( \theta_0 \) were chosen in order to show different possibilities for the boundary continuum. Figures 2 and 3 illustrate boundaries that are probably indecomposable continua, Figs. 4–8 illustrate boundaries that contain “bubbles”, i.e., invariant open sets, and Fig. 9 illustrates the simplest sort of boundary, one that is homeomorphic to a straight line. We show in this paper that the cases illustrated by Figs. 2, 3, and 9 happen. It follows from KAM theory that the invariant open sets occur for appropriate parameter values. We hope to investigate these open sets from a purely topological point of view in a future paper.

**Fig. 2.** The figure (probably) illustrates the situation in Theorem 12. We take \( \lambda = 5 \) and \( \theta_0 = 3 \), so that \( F = J_5 \circ R_3 \). The squeeze factor \( \lambda \) dominates here. The fluids and resulting boundary are pictured at the top, while the lower figures give a closer look at the boundary. The figure at the middle right is a blow-up of the portion of the figure at the middle left inside the small rectangle. Likewise, the lower left is a blow-up of the portion of the figure at the middle right inside the rectangle, and the figure at the lower right is a blow-up of the portion of the lower left figure inside the small rectangle. Shaded regions are part of the red fluid, but portions of the red have been erased or lightened in order to show the boundary more clearly. The boundary is at the edge where black and white meet.
Fig. 3. The figure (probably) illustrates the situation in Theorem 13. We take \( \lambda = 20 \) and \( \theta_0 = 12 \), so that \( F = J_{20} \circ R_{12} \). The squeeze factor \( \lambda \) dominates here, too, but there is quite a bit more rotation than in the previous example. The fluids and resulting boundary are pictured at the top, while the lower figures give a closer look at the boundary. The figure at the middle right is a blow-up of the portion of the figure at the middle left inside the small rectangle. Likewise, the lower left is a blow-up of the portion of the figure at the middle right inside the rectangle, and the figure at the lower right is a blow-up of the portion of the lower left figure inside the small rectangle. Shaded regions are part of the red fluid, but portions of the red have been erased or lightened in order to show the boundary more clearly. The boundary is at the edge where black and white meet.
Fig. 4. In this figure the situation is quite different from the previous ones. We take $\lambda = 2$ and 
$\theta_0 = 24.5$, so that $F = J_2 \circ R_{24.5}$. The squeeze factor $\lambda$ is dominated here by the rotation, which is quite strong. The fluids and resulting boundary are pictured at the top left. (The $x$-scale runs from $-20$ to $20$.) The other figures give a closer look at the boundary. The figure at the top right is a blow-up of the portion of the figure at the top left inside the small rectangle. Likewise, the lower left is a blow-up of the portion of the figure at the top right inside the rectangle. The figure at the lower right is a blow-up of the portion of the lower left figure inside the smaller rectangle, but inside the "bubble", additionally, a number of point trajectories have been plotted. Note the typical or Hamiltonian sort of structure demonstrated by these trajectories. It appears in this figure that there is a smaller invariant open region which has $6$ components. (Trajectories appear to remain inside these regions, too, with the $6$ smaller regions being "periodic" of period $6$.) These bubbles are an important feature of stirring processes and often a big problem when they occur in physical systems. In our model, they apparently are fairly rare. (See the section on random results, as well as the periodic section.)
Fig. 5. This is another bubble picture. We take $\lambda = 2$ and $\theta_0 = 3$, so that $F = J_2 \circ R_3$. The squeeze factor $\lambda$ is dominated here somewhat by the rotation, but the rotation is not nearly so strong as in the previous example. The fluids and resulting boundary are pictured in the picture at the top, top left. (The $x$-scale runs from $-5$ to $5$.) The other figures in the top picture give a closer look at the boundary. The figure at the top, top right is a blow-up of the portion of the figure at the top, top left inside the small rectangle. Likewise, the top, lower left is a blow-up of the portion of the figure at the top, top right inside the rectangle. The figure at the top, lower right is the same as the top, lower left figure, but inside the period two bubbles, additionally, a number of point trajectories have been plotted. Note the typical or Hamiltonian sort of structure demonstrated by these trajectories, as well as those trajectories plotted in the bottom picture. The middle and bottom pictures depict a further blow-up of the invariant region. It appears that the invariant open set has two components here, which flip back and forth as the map is iterated. (The rotation is approximately $\pi$ radians.) The bottom picture is just the middle picture with trajectories plotted in addition to the boundary. The origin is a saddle point here, and one trajectory appears to be a separatrix (the curve that appears to be folding onto a curve containing the origin).
Fig. 6. This is another bubble situation, but it shows what occurs when the rotation strongly dominates a mild squeeze factor. We take $\lambda = 1.5$ and $\theta_0 = 15$, so that $F = J_{1.5} \circ R_{15}$. The fluids and resulting boundary are pictured in the picture at top. (The $x$-scale runs from $-1$ to $1$.) The figure in the middle is a blow-up of the top figure, and it seems to show a rather large invariant open region containing the origin with a smaller invariant open region with four components around the large one. The bottom figure is the same as the middle figure, but inside the bubbles a number of point trajectories have been plotted.
Fig. 7. Now we examine a mild rotation and a mild squeeze. We take $\lambda = 2$ and $\theta_0 = 1$, so that $F = J_2 \circ R_1$. (The $x$-scale runs from $-1$ to $1$ in the top, top left picture, and it runs from $-3$ to $3$ in the bottom, top left picture.) The other figures in the top picture give successively closer looks at a portion of the boundary. There is a fixed point at both the "pile up" places in the top pictures. This occurs because $\arctan(2) - \arctan(1/2) < 1$ (Lemma 14). Theorems 17 and 18 also suggest that this fixed point is a saddle point. The figure at the bottom shows more features of this boundary. The bottom, top right, is a blow-up of the portion of the figure at the bottom, top left, inside the small rectangle. The bottom, lower left shows the basins of infinity of the figure at the top right. That is, it shows those points that go to either $\infty^+$ or to $\infty^-$ under iteration by $F$. The different bands show successively higher times in order to leave the unit disk. The figure at the bottom, lower right, is the same as the bottom, lower left, figure, but inside the bubble, additionally, a number of point trajectories have been plotted.
Fig. 8. Now we examine a mild rotation and a somewhat stronger squeeze. We take $\lambda = 3$ and $\theta_0 = 1$, so that $F = J_3 \circ R_1$. (The $x$-scale runs from $-1$ to $1$ in the top, top left picture.) There is again a fixed point at both the corners in the pictures, which occurs because $\arctan(3) - \arctan(1/3) < 1$ (Lemma 14). The figure at the bottom left shows a blow-up of the boundary, and is centered at the origin with $x$-scale from $-0.2$ to $0.2$. The top right picture shows the basins of infinity of the figure at the top left. That is, it shows those points that go to either $\infty^+$ or to $\infty^-$ under iteration by $F$. The different bands show successively higher times in order to leave the unit disk. The figure at the lower right is a blow-up of the upper right figure, but inside the bubble a number of point trajectories have been plotted. In this figure we are close to the situation of the next figure, but are not quite there. It is important that $\arctan(3) - \arctan(1/3) < 1$, even though it is just barely that. When $\arctan(\lambda) - \arctan(1/\lambda) \geq 1$, Theorem 16 applies, and the boundary must be much simpler.
Fig. 9. Last we examine an extremely mild rotation and a somewhat stronger squeeze. We take $\lambda = 3$ and $\theta_0 = 0.7$, so that $F = J_3 \circ R_{0.7}$. (The $x$-scale runs from $-1$ to $1$ in the top left picture.) There is now no fixed point except at the origin in the pictures, which occurs because $\arctan(3) - \arctan(1/3) > 0.7$ (Lemma 14). The figure at the top right shows a blow-up of the boundary, and is centered at the origin with $x$-scale from $-0.2$ to $0.2$. The origin is marked in this picture with a cross. The bottom left picture shows the basins of infinity of the figure at the top right. The different bands show successively higher times in order to leave the unit disk. The figure at the lower right essentially shows the unstable and stable manifolds of the origin, which is a saddle point in this case. When $\arctan(\lambda) - \arctan(1/\lambda) \geq 1$, Theorem 16 applies, and the boundary must be homeomorphic to the real line. The dynamics here are extremely simple, and this is not a very interesting case. The unstable and stable manifolds were obtained by iterating the axes under $F$ and its inverse, respectively. The stable manifold is also $\tilde{A}^-$. (The set $\tilde{A}^-$ is defined in Proposition 1 and discussed in Sections 3 and 4.)
2. Background, definitions, and notation

We take as our space the plane $\mathbb{R}^2$, and for convenience, endow the plane with a partial compactification. Let $\infty = \{ \infty^+_x, \infty^+_y, \infty^-_x, \infty^-_y \}$, and compactify the $x$- and $y$-axes with these four new points, so that the resulting space $Y = \mathbb{R}^2 \cup \infty$ is homeomorphic to the subspace

$$X = \{ x \in \mathbb{R}^2 : d(x, \mathbf{0}) < 1 \text{ or } x \in \{ (-1, 0), (1, 0), (0, -1), (0, 1) \} \}$$

of $\mathbb{R}^2$ endowed with the usual subspace topology. Let

$$\mathbb{R}_x = \{ (x, 0) : x \in \mathbb{R}^2 \} \cup \{ \infty^+_x, \infty^-_x \},$$
$$\mathbb{R}_y = \{ (0, y) : y \in \mathbb{R}^2 \} \cup \{ \infty^+_y, \infty^-_y \}.$$  

(As the notation suggests,

$$\lim_{x \to \infty} (x, 0) = \infty^+_x, \quad \lim_{x \to -\infty} (x, 0) = \infty^-_x,$$

$$\lim_{y \to \infty} (0, y) = \infty^+_y, \quad \lim_{y \to -\infty} (0, y) = \infty^-_y.$$  

The fluid motion permitted in our model consists of two invertible, area preserving motions that are alternated. One is the squeeze map $J_x$ on $Y$: suppose $X > 1$; for $(x, y)$ in $\mathbb{R}^2$, define the squeeze map $J_x(x, y) = (x, y/x)$, and, for $l \in \infty$, define $J_x(l) = l$.

This map represents fluid entering the container (coming from $\infty^+_x$ at time $-\infty$ if the fluid is red or $\infty^-_x$ if the fluid is blue), and leaving the container (going to $\infty^+_x$ at time $\infty$ if the fluid is red or $\infty^-_x$ if the fluid is blue). See Fig. 10. The other map is the stir or rotation map $R_{\theta_0}$ on $Y$. Suppose $\theta_0 > 0$. Using polar coordinates, define the map $R_{\theta_0}$ so that there is no motion outside the unit circle centered at the origin, i.e.,

$$R_{\theta_0}(r, \theta) = (r, \theta), \text{ if } r > 1. \text{ Define } R_{\theta_0}^-(r, \theta) \text{ for points inside or on the unit circle } 0 \leq r \leq 1 \text{ so that each circle concentric about the origin is invariant, and so that if } S \text{ is one of these circles, then } R_{\theta_0}^+|S \text{ is a simple counterclockwise rotation. Further, define } R_{\theta_0} \text{ so that it is a homeomorphism, so that the limit as } r \text{ approaches 0 of the rotation is } \theta_0, \text{ and so that circle rotation strictly decreases from } \theta_0 \text{ to 0 as } r \text{ goes from 0 to 1. See Fig. 11. For convenience, we make the following definition: when speaking of the map } R_{\theta_0}, \text{ the statement that the origin is rotated by } \theta_0 \text{ radians means that the limit as } r \text{ approaches 0 of the rotation is } \theta_0. \text{ The specific form of } R_{\theta_0} \text{ is not usually important to us, but many examples of such homeomorphisms exist. For example, we can define } R_{\theta_0}(r, \theta) = (r, \theta) \text{ for } (r, \theta) \text{ not in the open unit ball } D_1(0), \text{ and } R_{\theta_0}(r, \theta) = (1-r^2)^2 \theta_0 + \theta\text{ for } (r, \theta) \in D_1(0). \text{ This example, in addition to satisfying the properties required, is also a diffeomorphism, and it is the map used in making the computer generated figures. (See Figs. 2–9.) It is possible to choose a rotation map with the properties desired so that it is a } C^\infty \text{ diffeomorphism. Note that the specific map } R_{\theta_0} \text{ defined above expressed in rectangular coordinates is } R_{\theta_0}(x, y) = (x, y) \text{ for } (x, y) \notin D_1(0) \text{ and } R_{\theta_0}(x, y) = (x', y'), \text{ where}

$$x' = x \cos ((1 - x^2 - y^2)^2 \theta_0) - y \sin ((1 - x^2 - y^2)^2 \theta_0),$$
$$y' = y \cos ((1 - x^2 - y^2)^2 \theta_0) + x \sin ((1 - x^2 - y^2)^2 \theta_0)$$

for $(x, y) \in D_1(0)$.}
Fig. 10. The figure illustrates the simple dynamics of $J_\lambda$ (acting alone). See also Trivial Example 2.

Fig. 11. The figure illustrates the simple dynamics of $R_{\theta_0}$ (acting alone). See also Trivial Example 3. Note that as $r$ goes from 0 to 1, rotation decreases from $\theta_0$ to 0.

**Remark.**

(1) We emphasize that both $J_\lambda$ and $R_{\theta_0}$ are area preserving where area is computed in the standard way in the plane and the points of $\infty$ have area 0.

(2) The points of $\infty \cup \{0\}$ are all fixed points for both $J_\lambda$ and $R_{\theta_0}$. Additionally, $R_x$ and $R_y$ are invariant under $J_\lambda$, and each point not in $D_1(0) = \{(x, y): d(x, 0) < 1\}$ is fixed under $R_{\theta_0}$. For $0 \leq r < 1$, let $S_r = \{x \in \mathbb{R}^2: d(x, 0) = r\}$. Each circle $S_r$ is invariant under $R_{\theta_0}$. Similarly, for $r > 0$, let $D_r(0) = \{(x, y): d(x, 0) < r\}$.

Use $\mathbb{N}$ to denote the positive integers, $\bar{\mathbb{N}}$ to denote the nonnegative integers, and $\mathbb{Z}$ to denote the integers. We use $\textbf{I}$, $\textbf{II}$, $\textbf{III}$, and $\textbf{IV}$ to denote the four (open) quadrants of the plane in the usual fashion. Let $\pi_x : Y \to \mathbb{R}_x$ and $\pi_y : Y \to \mathbb{R}_y$ denote the respective projection maps. Let $\|x\|$ denote the usual norm of points in $\mathbb{R}^2$, i.e., if $x = (x_1, x_2)$, then $\|x\| = \sqrt{x_1^2 + x_2^2}$. If $W \subseteq Y$, then $W^o$ denotes the interior of $W$ in $Y$, while $\bar{W}$ denotes the closure of $W$ in $Y$.

A **continuum** is a compact, connected, metric space. If $X$ is a continuum, and $X'$ is a closed, connected subset of $X$, then $X'$ is a **subcontinuum** of $X$. A continuum is **decomposable** if it can be written as the union of two proper subcontinua. Otherwise it is **indecomposable**. Equivalently, a continuum $X$ is indecomposable if and only if each proper subcontinuum of $X$ is nowhere dense in $X$. Note that indecomposable continua cannot be locally connected at any point.
All indecomposable continua share a certain structure. If \( X \) is a continuum and \( x \) is a point in \( X \), then the **composant** of \( X \) containing \( x \) is the set \( \text{Com}(x) = \{ y \in X : \text{there is a proper subcontinuum } X' \text{ of } X \text{ containing both } x \text{ and } y \} \). Further, \( \text{Com}(x) \) is a dense, connected subset of \( X \). All indecomposable continua have \( c \) many disjoint composants. The composants of an indecomposable continuum partition that continuum, i.e., if \( \text{Com}(x) \cap \text{Com}(y) \neq \emptyset \), then \( \text{Com}(x) = \text{Com}(y) \). Each composant of an indecomposable continuum is a first category, \( F_\sigma \)-set in that continuum. (An \( F_\sigma \)-set is one that can be written as the countable union of nowhere dense sets.) If a continuum contains two disjoint composants, then it must be indecomposable. For more information about the structure of indecomposable continua, see the second volume of C. Kuratowski's *Topology* [4].

Suppose that for some \( \lambda > 1, \theta_0 > 0, J = J_\lambda \) and \( R = R_{\theta_0} \). For each integer \( i \), let \( F_i \) denote one of the maps \( J \) and \( R \). Now when our topological observer looks at the stirring process at time \( t = 0 \), he/she is able to tell what map was applied at time \(-1\) fairly easily, what map was applied at time \( t = -2 \) not quite so easily—in general, the further back in time the map was applied (large negative \( i \)), the more difficult it is to tell whether the map applied was \( J \) or \( R \). The phenomena present at time 0 are the result of all maps applied since "time \( t = -\infty \)"; i.e., these phenomena are a result of the limiting process \( \lim_{i \to -\infty} F_{-1} \circ F_{-2} \circ F_{-3} \circ \cdots \circ F_{-i} \). When we say that the process has been going on for all time in the past, we mean that we are considering this limiting process. Note that

\[
(F_{-1} \circ F_{-2} \circ F_{-3} \circ \cdots \circ F_{-i})^{-1} = F_{-i}^{-1} \circ \cdots \circ F_{-3}^{-1} \circ F_{-2}^{-1} \circ F_{-1}^{-1}.
\]

A finite collection \( C = \{c_0, c_1, \ldots, c_n\} \) is a **chain** if \( c_i \cap c_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). If the elements of \( C \) are open sets, then \( C \) is an **open chain**. However, in this manuscript our chains have elements consisting of closed neighborhoods intersecting only at their boundaries. Thus, we define a **tiling chain** \( C = \{c_0, c_1, \ldots, c_n\} \) to be a chain whose elements are closed sets and such that \( c_i^\circ \cap c_j^\circ = \emptyset \) for \( i \neq j \). The **mesh** of a chain \( C = \{c_0, c_1, \ldots, c_n\} \) is the positive number \( \sigma = \sup\{\text{diam}(c_i)\} \). If for each \( \varepsilon > 0 \), the continuum \( X \) has an open or tiling cover of mesh less than \( \varepsilon \), then \( X \) is an **arclike**, or **chainable**, or **snakelike** continuum. Equivalently, an arclike continuum \( X \) is one that admits for each \( \varepsilon > 0 \), a continuous map \( f_\varepsilon : X \to [0, 1] \) such that for each \( x \) in \([0, 1]\), \( \text{diam} f_\varepsilon^{-1}(x) \) is less than \( \varepsilon \).

Finally, let \( A_0 = [-1, 1] \times [-1, 1] \), and, in general, for each integer \( i \) define \( A_i = [-\lambda^i, \lambda^i] \times [-\lambda^{-i}, \lambda^{-i}] \). Let

\[
A = A^+ = \left( \bigcup_{j \in \mathbb{N}} A_j \right) \cup \{\infty_x^+, \infty_y^+\}, \quad \text{and} \quad A^- = \left( \bigcup_{j \in \mathbb{N}} A_{-j} \right) \cup \{\infty_x^-, \infty_y^-\}.
\]

The Knaster continua are the simplest indecomposable continua, and may be defined in association with the Smale horseshoe map as follows:

Suppose \( n \) is a positive integer greater than 1, and let \( D \) denote the stadium shaped region pictured in Fig. 12. Note that \( D \) is the union of two semicircular sets \( A \) and \( B \) and a rectangular region \( R \). (Warning: the semicircular set \( A \) in this horseshoe map—
Fig. 12. At the top, the stadium region $D$ is pictured. In the middle figure, $F_3(D)$ is shown in $D$, then $F_3^2(D)$ in $F_3(D)$ is indicated. At the bottom, $F_5(D)$ is shown in $D$, then $F_5^2(D)$ is indicated in $F_5(D)$. Note that $F_3$ involves folding $D$ inside itself 3 times, while $F_5$ involves folding $D$ inside itself 5 times.

Knaster continuum discussion is not the same as the planar sets $A, A^+, A^-, A_j (j \in \mathbb{Z})$ which are associated with our stirring model construction and are defined in the preceding paragraph. In later discussion and proofs, the set $B$ (with various superscripts, subscripts, and embellishments) also appears, and again these later defined $B$-sets are different from this semicircular set associated with the horseshoe map—Knaster continuum construction.) Then the horseshoe map $F_n$ takes $D$ into itself by contracting $R$ vertically, expanding $R$ horizontally, contracting $A$ and $B$ radially to fit on the now long and thin acted upon $R$, folding the resulting set $n - 1$ times, and reinserting it inside $D$. This can be done so that $F_n$ is a diffeomorphism, $F_n(D) \cap R$ consists of $n$ rectangles, each vertical line segment in each of these rectangles is the image of a vertical line segment in $D$ under $F_n$, each horizontal line segment in each of these rectangles is the image of a horizontal line segment in $D$, $F_n(A)$ is contained in the interior of $A$ and $F_n(B)$ is contained in the interior of $B$. It is well known that the dynamical system $(D, F_n)$ has the following properties [1,2,3,6] (see Fig. 12):

1. Since $D \supseteq F_n(D)$, $K_n = \bigcap_{m \geq 0} F_n^m(D)$ is the intersection of a nested collection of continua, and is therefore a continuum. The continuum $K_n$ is arclike and indecomposable, and each of its proper subcontinua is an arc. Also, $F_n(K_n) = K_n$. 

(2) The set \( C_n = \bigcap_{m \in \mathbb{Z}} F_n^m(R) \) is a Cantor set, and \( F_n(C_n) = C_n \).

(3) For \( n \) even, \( \bigcap_{m \geq 0} F_n^m(A) \) consists of a single point \( p_n \), which is an attracting fixed point. The basin of attraction for \( p_n \) (i.e., the collection of all points in \( D \) attracted to \( p_n \)) consists of all points \( x \) in \( D \) not on a vertical line intersecting \( C_n \).

(4) For \( n \) odd, \( \bigcap_{m \geq 0} F_n^m(A) \) consists of a single point \( p_n \), and \( \bigcap_{m \geq 0} F_n^m(B) \) consists of a single point \( q_n \), with both \( p_n \) and \( q_n \) being attracting fixed points. The union of the basins of attraction for the points \( p_n \) and \( q_n \) is the set of all points \( x \) in \( D \) not on a vertical line intersecting \( C_n \).

(5) The homeomorphism \( F_n \mid C_n \) is conjugate to the full \( n \)-shift \( \sigma_n \) on \( \Sigma_n \).

3. Basic results

**Proposition 1.** For \( \lambda > 1 \) and \( \theta_0 > 0 \), \( J_\lambda(A) \subseteq A \) and \( R_{\theta_0}(A) \subseteq A \). If for each \( i \in \mathbb{N} \), \( F_i = J_\lambda \) or \( F_i = R_{\theta_0} \), then

\[
\tilde{A} = \bigcap_{j=0}^{\infty} F_j \circ F_{j-1} \circ \cdots \circ F_0(A)
\]

is a continuum contained in \( A \), and, likewise,

\[
\tilde{A}^- = \bigcap_{j=0}^{\infty} \left( F_j \circ F_{j-1} \circ \cdots \circ F_0 \right)^{-1}(A)
\]

is a continuum.

**Proof.** Note that

\[
J_\lambda(A) = \left( \bigcup_{j=1}^{\infty} [-\lambda^{-j}, \lambda^{-j}] \times [-\lambda^{-j}, \lambda^{-j}] \right) \cup \{ \infty^+, \infty^- \} \subseteq A,
\]

and \( R_{\theta_0}(A_0) = A_0 \), while for \( x \notin A_0 \), \( R_{\theta_0}(x) = x \). The result follows since \( A \) is compact. Similar arguments prove that

\[
\tilde{A}^- = \bigcap_{j=0}^{\infty} \left( F_j \circ F_{j-1} \circ \cdots \circ F_0 \right)^{-1}(A)
\]

is a continuum. \( \square \)

**Trivial Example 2.** For \( \lambda > 1 \), \( \bigcap_{i=0}^{\infty} J_\lambda^i(A) = \mathbb{R} \). Further, if \( z = (x, y) \in \mathbb{R}^2 \), and \( y > 0 \), then each \( \pi_y J_\lambda^i(z) > 0 \) for each \( i \), and \( \lim_{i \to \infty} J_\lambda^i(z) = \infty^+ \) if \( x > 0 \), \( \lim_{i \to \infty} J_\lambda^i(z) = \infty^- \) if \( x < 0 \), and \( \lim_{i \to \infty} J_\lambda^i(z) = 0 \) if \( x = 0 \). Likewise, if \( z = (x, y) \in \mathbb{R}^2 \), and \( y < 0 \), then each \( \pi_y J_\lambda^i(z) < 0 \) for each \( i \), and \( \lim_{i \to \infty} J_\lambda^i(z) = \infty^+ \) if \( x > 0 \), \( \lim_{i \to \infty} J_\lambda^i(z) = \infty^- \) if \( x < 0 \), and \( \lim_{i \to \infty} J_\lambda^i(z) = 0 \) if \( x = 0 \). If \( z = (x, 0) \in \mathbb{R}^2 \), and \( x > 0 \), then \( \pi_y J_\lambda^i(z) = 0 \) for each \( i \), and \( \lim_{i \to \infty} J_\lambda^i(z) = \infty^+ \) if \( x > 0 \), \( \lim_{i \to \infty} J_\lambda^i(z) = \infty^- \) if \( x < 0 \), and \( \lim_{i \to \infty} J_\lambda^i(z) = 0 - J_\lambda^i(z) \) if \( x = 0 \). (See Fig. 10.)
Trivial Example 3. For \( \theta_0 > 0 \), \( \bigcap_{n=0}^{\infty} R_{\theta_0}^n(A) = A \). For \( z \notin D_1(0) \), \( R_{\theta_0}(z) = z \) and \( R_{\theta_0}(0) = 0 \). For \( z \in D_1(0) \setminus \{0\} \), \( d(z,0) = r_z \) for some \( r_z > 0 \). If \( S_{r_z} \) undergoes an irrational rotation, then each member of \( S_{r_z} \) is a limit of some subsequence of \( R_{\theta_0}(z), R_{\theta_0}^2(z), R_{\theta_0}^3(z), \ldots \), and if \( S_{r_z} \) undergoes a rational rotation, then \( z \) is periodic of period \( q_z \), where for some positive integer \( p_z \), the rotation of \( S_{r_z} \) is \( (2\pi)p_z/q_z \). (See Fig. 1.) Also, if \( z \in D_1(0) \), then there exist sequences \( n_1, n_2, \ldots \) and \( m_1, m_2, \ldots \) of positive integers and sequences \( z_1, z_2, \ldots \) and \( w_1, w_2, \ldots \) of points of \( D_1(0) \) such that

1. \( \pi_y(z_i) > 0 \) for each \( i \) and \( \pi_y(w_i) < 0 \) for each \( i \), and
2. \( \lim_{i \to \infty} R_{\theta_0}^{n_i}(z_i) = z = \lim_{i \to \infty} R_{\theta_0}^{m_i}(w_i) \).

Proposition 4. For \( \lambda > 1 \), \( 0 < r \leq 1 \), \( J_{\lambda}(S_r) = F_r \) is an ellipse, as is \( J_{-\lambda}^{-1}(S_r) = F_{-r} \). Further, \( J_{\lambda} \) takes vertical line segments to vertical line segments, horizontal line segments to horizontal line segments, lines to the origin to lines through the origin, lines to lines, and for each \( r \), \( J_{\lambda}(D_r(0)) \cap J_{-\lambda}^{-1}(D_r(0)) \subseteq D_r(0) \).

Suppose that \( X \) is a metric space, and \( f : X \to X \) is a homeomorphism on \( X \). Then the point \( x \) in \( X \) is a wandering point for \( f \) if there exists some open set \( o \) in \( X \) which contains \( x \) and has the property that \( f^n(o) \cap f^m(o) = \emptyset \) for each \( n, m \in \mathbb{Z}, n \neq m \).

Proposition 5. If \( F \) is a finite composition of the maps \( J_{\lambda} \) and \( R_{\theta_0} \) with \( F \) not \( R_{\theta_0}^n \) for some \( n > 0 \), then \( Y \setminus (A_0 \cup \infty) \) consists of wandering points and \( \tilde{A} = \bigcap_{m \geq 0} F^m(A) \) is a continuum with \( (\tilde{A} \setminus A_0)^o = \emptyset \).

Proof. That \( \tilde{A} \) is a continuum follows from Proposition 1. Since \( F \mid (Y \setminus A_0) \) is a finite composition of \( J_{\lambda} \)'s, \( Y \setminus (A_0 \cup \infty) \) consists of wandering points. Suppose that the proposition is not true. Specifically, suppose \( v \) is an open set in \( \tilde{A} \setminus A_0 \). Without loss of generality, suppose \( v \subseteq A_i \setminus A_{i-1} \) for some \( i > 0 \). Then \( F^i(v) = J_{\lambda}^{n_0}(v) \) for some \( n_0 \geq 1 \), since \( R_{\theta_0}(x) = x \) for \( x \) not in \( A_0 \). Further, \( \{F^n(v) : n \geq 1\} \) is a mutually disjoint collection of open sets, which implies that \( \{F^n(v) : n \in \mathbb{Z}\} \) is a mutually disjoint collection. Now \( F(\tilde{A}) = F(\bigcap_{m \geq 0} F^m(A)) = \tilde{A} \), so \( \bigcup\{F^n(v) : n \in \mathbb{Z}\} \subseteq \tilde{A} \subseteq A \), and there is some \( N \) such that if \( n \geq N \), \( F^{-n}(v) \subseteq A_0 \). Since each factor map of \( F \) is area preserving, \( F \) is area preserving, so \( F^{-n}(v) \) has the same positive measure, say \( \alpha \), as \( v \) does. This is a contradiction, since this means that \( A_0 \) contains an infinite collection of disjoint open sets, all of measure \( \alpha \). The proposition follows. \( \square \)

Lemma 6. If \( F \) is a finite composition of the maps \( J_{\lambda} \) and \( R_{\theta_0} \), then no point of \( (\tilde{A} \cap A_0)^o \) wanders. If \( C \) is a component of \( (\tilde{A} \cap A_0)^o \), then \( F^n(C) = C \) for some \( n \).

Proof. If \( x \) is a point of \( (\tilde{A} \cap A_0)^o \) which is wandering, then there is some open set \( v \) containing \( x \) and contained in \( (\tilde{A} \cap A_0)^o \) with the property that \( \{F^n(v) : n \in \mathbb{N}\} \) is a collection of mutually disjoint open sets. But \( v \) has some positive measure \( \alpha \), and each member of \( \{F^{-n}(v) : n \in \mathbb{N}\} \) has measure \( \alpha \) and is contained in \( A_0 \). This is a contradiction.
Fig. 13. The regions $M_1, M_2, M_3$ and $M_4$, as well as the lines $L_\lambda^+, L_\lambda^-, L_{1/\lambda}^+$ and $L_{1/\lambda}^-$ are pictured. Points on $L_\lambda^+$ and $L_\lambda^-$ stay the same distance from 0 when mapped by $J_\lambda$. Those points on $L_\lambda^+$ map to $L_{1/\lambda}^+$, while those points on $L_\lambda^-$ map to $L_{1/\lambda}^-$. Points in $M_2$ and $M_4$ map closer to 0 by $J_\lambda$ while points in $M_1$ and $M_3$ map further away from 0.

If $(\tilde{A} \cap A_0)^0 \neq \emptyset$, then $F(\tilde{A} \cap A_0)^0 = (\tilde{A} \cap A_0)^0$. Let $C$ denote the collection of all components of $(\tilde{A} \cap A_0)^0$ (which is a collection of nonempty disjoint open sets). If $C \in C$, then for some positive integer $n$, $F^n(C) = C$: for some $n$, $F^n(C) \cap C \neq \emptyset$, for otherwise there again exists an infinite collection of mutually disjoint open sets each with positive measure and each contained in $A$. If $F^n(C) \cap C \neq \emptyset$, then $F^n(C) \cap C = C$, for otherwise $C$ is not a component of $C$. 

The following sets come up in many of the remaining results. Thus, we define

$$M_1 = \{(x, y): |y| < \lambda x\}, \quad M_2 = \{(x, y): y > \lambda |x|\},$$

$$M_3 = \{(x, y): |y| < -\lambda x\}, \quad M_4 = \{(x, y): -y > \lambda |x|\}.$$ 

Similarly, define

$$M_1^- = \{(x, y): |y| < x / \lambda\}, \quad M_2^- = \{(x, y): y > |x| / \lambda\},$$

$$M_3^- = \{(x, y): |y| < -x / \lambda\}, \quad M_4^- = \{(x, y): -y > |x| / \lambda\}.$$ 

Also define the boundary lines for these sets:

$$L_\lambda^+ = \{(x, \lambda x): x \in \mathbb{R}\}, \quad L_\lambda^- = \{(x, -\lambda x): x \in \mathbb{R}\},$$

$$L_{1/\lambda}^+ = \{(x, x / \lambda): x \in \mathbb{R}\}, \quad L_{1/\lambda}^- = \{(x, -x / \lambda): x \in \mathbb{R}\}.$$ 

Lemma 7. Suppose $\lambda > 1$. Then $Y \setminus (L_{1}^+ \cup L_{-1}^-)$ consists of the four mutually disjoint regions $M_1, M_2, M_3,$ and $M_4$. (See Fig. 13.) If $(x, y)$ is in $M_1 \cup M_3$, then $d((x, y), 0) < d(J_\lambda(x, y), 0)$; if $(x, y)$ is in $M_2 \cup M_4$, then $d((x, y), 0) > d(J_\lambda(x, y), 0)$; and if $(x, y)$ is in $L_\lambda^+ \cup L_\lambda^-$, then $d((x, y), 0) = d(J_\lambda(x, y), 0)$ with $J_\lambda(x, y)$ in $L_{1/\lambda}^+ = \{(x, x / \lambda): x \in \mathbb{R}\}$ if $(x, y)$ is in $L_\lambda^+$, and with $J_\lambda(x, y)$ in $L_{1/\lambda}^- = \{(x, -x / \lambda): x \in \mathbb{R}\}$ if $(x, y)$ is in $L_{1/\lambda}^-$. Further, if $(x, y) \in S_r \cap (L_\lambda^+ \cup L_{1/\lambda}^-)$, then $J_\lambda(x, y) \in S_r$.
Lemma 8. Let $F = R_{\theta_0} J_{\lambda}$ for some $\theta_0 > 0$, $\lambda > 1$. For each $0 < m < \lambda$, let

$$W_m = \{(x_1, y_1): x_1 \in \mathbb{R} \text{ and } |y_1| \leq m\}.$$

If $x$ is a point in $Y$, then the forward orbit $O^+_F(x) = \{F^n(x): n \geq 0\}$ is not contained in $D_1(0) \cap W_m$. More specifically, if $x$ is in $W_m$, there is some $N > 0$ such that if $n > N$, then (each) $F^n(x)$ is in $M_1 \setminus D_1(0)$, or (each) $F^n(x)$ is in $M_3 \setminus D_1(0)$. Similarly, if, for each $m > \lambda$, $W_m^+ = \{(x_1, y_1): x_1 \in \mathbb{R} \text{ and } |y_1| \geq m\}$, and $x$ is a point in $Y$ whose forward orbit $O^+_F(x) = \{F^n(x): n \geq 0\}$ is contained in $D_1(0) \cap W_m^+$, then the sequence $\langle F^n(x) \rangle_{n \geq 0}$ converges to the point $0$.

Proof. Suppose $x$ is a point in $D_1(0) \cap W_m$, and $O^+_F(x)$ is contained in $D_1(0) \cap W_m$. Let $R_{\theta_0} - R$ and $J_{\lambda} - J$, so that $F = RJ$. Then

$$d(x, 0) < d(J(x), 0) = d(F(x), 0) < d(JF(x), 0) = d(F^2(x), 0) < \cdots.$$ 

In addition, $J(x)$ is closer to the $x$-axis than was $x$. Thus, $J(x)$ is on a circle $S_{a(x)}$ of larger radius than the circle $S_a$ that $x$ is on, and $R$ rotates $J(x)$ less than it would have rotated $x$. Suppose there is some $0 < a \leq 1$ such that the sequence $\langle d(F^n(x), 0) \rangle_{n \geq 0}$ converges to $a$. Since $D_1(0) \cap W_m$ is compact, we may assume that the sequence $\langle d(F^n(x), 0) \rangle_{n \geq 0}$ converges to some point $z$ in $D_1(0) \cap W_m$. (Note that the sequence $\langle d(F^n(x), 0) \rangle_{n \geq 0}$ is strictly increasing, and all accumulation points of $\langle F^n(x) \rangle_{n \geq 0}$ must be in $S_a$.) Further, since $J(W_m) \subseteq W_m^*$, $F(z)$ must be in $D_1(0) \cap W_m$. But then the sequence $\langle F^{n+1}(x) \rangle_{n \geq 0}$ converges to the point $F(z)$, so that $z = F(z)$, while $d(F(z), 0) > d(z, 0)$. Thus, there is a least $N$ such that $F^N(x)$ is not in $D_1(0)$, and $F^N(x)$ is either in $M_1$ or $F^N(x)$ is in $M_3$. If $F^N(x)$ is in $M_1$, then for all $n > N$, $F^n(x)$ is in $M_1$, since $F^{N+1}(x) = J^m F^N(x)$ for $m > 0$. A similar statement holds if $F^N(x)$ is in $M_3$. The proof that if $x$ is a point in $Y$ whose forward orbit $O^+_F(x) = \{F^n(x): n \geq 0\}$ is contained in $D_1(0) \cap W_m$, then the sequence $\langle F^n(x) \rangle_{n \geq 0}$ converges to the point $0$ is similar and we omit it. \Box

Corollary 9. Let $F = R_{\theta_0} J_{\lambda}$ for some $\theta_0 > 0$, $\lambda > 1$. If $o$ is a component of an invariant open set for $F$ contained in $A$, then it is not possible for $\{F^n(o): n \in \mathbb{Z}\}$ to be contained in some $W_m$, where $0 < m < \lambda$ and $W_m = \{(x_1, y_1): x_1 \in \mathbb{R} \text{ and } |y_1| \leq m\}$, nor is it possible for $\{F^n(o): n \in \mathbb{Z}\}$ to be contained in some $W_m$, where $m > \lambda$ and $W_m = \{(x_1, y_1): x_1 \in \mathbb{R} \text{ and } |y_1| \geq m\}$.

Lemma 10. Suppose $F = R_{\theta_0} J_{\lambda}$ for some $\theta_0 > 0$, $\lambda > 1$, and $f: [a, b] \to \mathbb{R}^2$ is a map such that, for $x$ in $[a, b]$, $\pi_x f(x) = x$. If $0 \leq x_1 < x_2$ implies $\|f(x_1)\| < \|f(x_2)\|$, then $\|J_f(x_1)\| < \|J_f(x_2)\|$ and $\|F(f(x_1))\| < \|F(f(x_2))\|$. If $x_1 < x_2 \leq 0$ implies $\|f(x_2)\| < \|f(x_1)\|$, then $\|J_f(x_2)\| < \|J_f(x_1)\|$ and $\|F(f(x_2))\| < \|F(f(x_1))\|$. 

Proof. Suppose first that $0 \leq x_1 < x_2$ implies $\|f(x_1)\| < \|f(x_2)\|$. Then let $f(x_1) = (x_1, y_1)$ and $f(x_2) = (x_2, y_2)$. Then $x_1^2 < x_2^2, x_1^2 + y_1^2 < x_2^2 + y_2^2$ and $x_2^2 + y_2^2 - (x_1^2 + y_1^2) > 0$. 


Thus \((x_2 - x_1) + (y_2 - y_1) > 0\) and, since \((x_2 - x_1) > 0\), \((y_2 - y_1)/\lambda > 0\), and \(\lambda^2(x_2 - x_1) + (y_2 - y_1)/\lambda^2 > 0\) or \(\|Jf(x_1)\| < \|Jf(x_2)\|\) and it follows, since \(R_{\theta_0}\) preserves norm, that \(\|F(f(x_1))\| < \|F(f(x_2))\|\).

Next suppose that \(x_1 < x_2 \leq 0\) implies \(\|f(x_2)\| < \|f(x_1)\|\). Then \(0 \leq -x_2 < -x_1\), and \(x_2^2 < x_1^2\). Hence, \((x_1^2 - x_2^2) > 0\) and \((y_1^2 - y_2^2) > 0\), so that \((x_1^2 - x_2^2) + (y_1^2 - y_2^2)/\lambda^2 > 0\) and \(\lambda^2(x_1^2 - x_2^2) + (y_1^2 - y_2^2)/\lambda^2 > 0\). Thus, \(\|Jf(x_2)\| < \|Jf(x_1)\|\) and \(\|F(f(x_2))\| < \|F(f(x_1))\|\). □

**Lemma 11.** Suppose \(\theta_0 > 0\) and \(R = R_{\theta_0}\) is a diffeomorphism. If \(\lambda\) is sufficiently large, \(\theta_0\) is not an odd multiple of \(\pi/2\), \(F = R\theta_0J\lambda\), and \(V = \{(x, y) : |x| \leq 1/\lambda\}\), then there exists a positive number \(M\) such that for each positive integer \(n\) and each \(a\) in \([-1, 1]\), each component of \(F^n(R_x \times \{a\}) \cap V\) is a differentiable curve which can be expressed as the graph of a differentiable function \(f : [-1/\lambda, 1/\lambda] \to [-1, 1]\) with derivative \(f'\) such that \(M > \sup\{|f'(x)| : x \text{ is in } [-1/\lambda, 1/\lambda]\}\) and \(4M^2/\lambda < 1\).

**Proof.** Now \(R(R_x) \cap R_y\) has a finite number of components. Choose \(\lambda > 0\) so that for each \(a\) in \([-1, 1]\), the set \(F(R_x \times \{a\}) \cap V\) has the same number of components as \(R(R_x) \cap R_y\) does. Since \(\theta_0\) is not an odd multiple of \(\pi/2\), this can be done, and although it may well be the case that the curves in \(F(R_x \times \{a\}) \cap V\) have vertical tangents at a finite number of places, none of these vertical tangents occurs on the \(y\)-axis, and \(V\) will not contain any such points on the curves. For each \(a\) each of those components can be expressed as the graph of a differentiable function \(f : [-1/\lambda, 1/\lambda] \to [-1, 1]\) and if 

\[
\mathcal{F} = \{f : [-1/\lambda, 1/\lambda] \to [-1, 1] : \text{ the graph of } f \text{ is the set } F(R_x \times \{a\}) \cap V \text{ for some } a\},
\]

then there is some positive number \(M\) such that

\[
M > \sup \{ |f'(x)| : x \text{ is in } [-1/\lambda, 1/\lambda] \text{ and } f \text{ is in } \mathcal{F} \}
\]

and \(M > 1\). Now, if necessary, increase the value of \(\lambda\) so that \(\lambda > 4M^2\). (Note that as the value of \(\lambda\) increases, the size of an appropriate \(M\) for that \(\lambda\) decreases because \(V\) gets narrower while \(|f'|\), for \(f\) in \(\mathcal{F}\), decreases. Thus, increasing \(\lambda\) does not increase an appropriate value of \(M\).) Also, there is \(0 < \varepsilon < 1\), such that if \(g : [-1, 1] \to [-1, 1]\) is a differentiable function with \(\sup \{ |g(x)| : x \text{ is in } [-1, 1]\} \leq \varepsilon\), \(\sup \{ |g'(x)| : x \text{ is in } [-1, 1]\} \leq \varepsilon\), \(\varepsilon < 1\), and \(\overline{g} : [-1/\lambda, 1/\lambda] \to [-1, 1]\) is a differentiable function whose graph is a component of \(\{R(x, g(x)) : x \text{ is in } [-1, 1]\}\) for some such differentiable function \(g\), then \(\sup \{ |\overline{g}'(x)| : x \text{ is in } [-1/\lambda, 1/\lambda]\} < 2M\). Again, if necessary, increase the value of \(\lambda\) so that \(2M^2/\lambda^2 < \varepsilon\). (This will not affect the properties just discussed. See Fig. 14. What we are really doing here is just using the squeeze of \(J\) to flatten out the rotated curves and put them "close" to the \(x\)-axis. Choosing \(\lambda\) and \(M\) large enough ensures this, and we also use this largeness to ensure that \(V\) is sufficiently narrow that the vertical tangencies are not included.)

Note that each component of each \(F(R_x \times \{a\}) \cap V\) extends from the left boundary of \(V\) to the right boundary of \(V\), and can be represented as the graph of a differentiable function
Fig. 14. The figure shows the relationship between $V = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1/\lambda\}$ and $R_{\theta_0} J_\lambda(A)$ necessary to ensure that Lemma 11 holds. (In this figure $\theta_0$ is approximately $\pi$).

$f: [-1/\lambda, 1/\lambda] \to [-1, 1]$ with derivative $f'$ such that $\sup\{|f'(x)| : x \in [-1/\lambda, 1/\lambda]\}$ is less than $M$. Then $Jf: [-1/\lambda, 1/\lambda] \to [-1, 1]$ is a differentiable curve with derivative $(Jf)'$ such that $\sup\{|(Jf)'(x)| : x \in [-1/\lambda, 1/\lambda]\}$ is less than $M/\lambda^2 < M^{-3}$, and $M/\lambda^2 < \varepsilon$. Then each component of $\{RJ(x, g(x)) : x \in [-1, 1]\}$ can be represented as the graph of a differentiable function $f_1: [-1/\lambda, 1/\lambda] \to [-1, 1]$ with derivative $f_1'$ such that $\sup\{|f_1'(x)| : x \in [-1/\lambda, 1/\lambda]\}$ is less than $2M$. It follows that

$$\left\{ (x, Jf_1(x)) : x \in [-1/\lambda, 1/\lambda] \right\} \subseteq \mathbb{R} \times [-1/\lambda, 1/\lambda],$$

and $\sup\{|(Jf_1)'(x)| : x \in [-1/\lambda, 1/\lambda]\} < M^{-3}$. Then each component of

$$\{RJ(x, f_1(x)) : x \in [-1/\lambda, 1/\lambda]\} \cap V$$

can be represented as the graph of a differentiable function $g_1: [-1/\lambda, 1/\lambda] \to [-1, 1]$ with $\sup\{|g_1'(x)| : x \in [-1/\lambda, 1/\lambda]\} < 2M$. This process can continue, and the result follows. \(\square\)

**Theorem 12.** $F = R_\pi J_\lambda$ and $R = R_\pi$ is a diffeomorphism. For sufficiently large $\lambda$, $\tilde{A} = \bigcap_{m \geq 0} F^m(A)$ has empty interior, and is homeomorphic to the Knaster continuum $K_3$. Also, $C = \{x \in \overline{D_1(0)} : F^n(x) \text{ is in } \overline{D_1(0)} \text{ for all } n\}$ is an invariant Cantor set, and $F|C$ is conjugate to a 3-shift. (See Fig. 2.)

**Proof.** It follows from Proposition 5 that $\tilde{A}\\setminus A_0$ has empty interior. Thus, we need only prove that $\tilde{A}\cap A_0$ has empty interior. If $(\tilde{A}\cap A_0)^o \neq \emptyset$, then $F(\tilde{A}\cap A_0)^o = (\tilde{A}\cap A_0)^o \subseteq A_0$. Let $u = (\tilde{A}\cap A_0)^o$. Choose $\lambda$ sufficiently large so that Lemma 11 holds for $F$. Let $V = \{(x, y) : |x| \leq 1/\lambda\}$. Note that $F(A_0) \cap V$ contains all points $z$ of $\tilde{A}$ such that $z$ and $F(z)$ are contained in $A_0$. Because each component of $F(A_0) \cap V$ extends from one vertical boundary of $V$ to the other, each component of $J[F(A_0) \cap V]$ extends from one boundary of $A_0$ to the other, and there are exactly three components of this set. (We are applying Lemma 11 here.) Thus, each component of $F[F(A_0) \cap V]$ intersects each component of $[F(A_0) \cap V]$ and extends from one vertical boundary of each component of $[F(A_0) \cap V]$ to the other. Hence, Lemma 11 guarantees that $F[F(A_0) \cap V] \cap [F(A_0) \cap V]$ consists of exactly nine
components, each extending from boundary to boundary and \( F[F(A_0) \cap V] \) consists of nine components also, and the process can continue. At the \( n \)th iterate of \( F \), \( F^n(A_0) \cap A_0 \) consists of exactly \( 3^{n-1} \) components, each extending from one boundary of \( A_0 \) to the other. Thus, it follows that \( \mathcal{C} = \bigcap_{n \geq 0} F^n(A_0) \) is a Cantor set of continua. Applying Lemma 11 and noting that if \( H \) is a vertical line and

\[
h_n = \max \{ ||\widetilde{H}|| : \widetilde{H} \text{ is a component of } H \cap (F^n(A_0) \cap A_0) \},
\]

then the size of \( \lambda \) forces \( \lim_{n \to \infty} h_n = 0 \), we also get that each component of \( \mathcal{C} \) is an arc and that \( \mathcal{C} = \{ x \in D_1(0) : F^n(x) \text{ is in } D_1(0) \text{ for all } n \} \) is an invariant Cantor set.

It follows from arguments quite similar to the standard horseshoe map arguments discussed in the second section that \( F \mid \mathcal{C} \) is conjugate to a 3-shift and that \( \tilde{A} \) is homeomorphic to the Knaster continuum \( K_3 \).

**Theorem 13.** Suppose \( \theta_0 > \pi/2, \theta_0 \) is not an odd multiple of \( \pi/2 \), and \( R = R_{\theta_0} \) is a diffeomorphism. If \( \lambda \) is sufficiently large, \( m \) is the unique positive odd integer such that \( (m - 2)\pi/2 < \theta_0 < m\pi/2 \), and \( F = R_{\theta_0}J_{\lambda} \), then \( \tilde{A} = \bigcap_{n \geq 0} F^n(A) \) has empty interior, and is homeomorphic to the Knaster continuum \( K_m \). Also,

\[
C = \{ x \in D_1(0) : F^n(x) \text{ is in } D_1(0) \text{ for all } n \}
\]

is an invariant Cantor set, and \( F \mid \mathcal{C} \) is conjugate to an \( m \)-shift. Further, \( m \) is the number of components of \( R(\mathbb{R}_x) \cap \mathbb{R}_y \). (See Fig. 3.)

**Proof.** It follows from Proposition 5 that \( \tilde{A} \setminus A_0 \) has empty interior. Thus, we need only prove that \( \tilde{A} \cap A_0 \) has empty interior. If \( (\tilde{A} \cap A_0)^\circ \neq \emptyset \), then \( F(\tilde{A} \cap A_0)^\circ = (\tilde{A} \cap A_0)^\circ \subseteq A_0 \). Let \( u = (\tilde{A} \cap A_0)^\circ \). Choose \( \lambda \) sufficiently large so that Lemma 11 holds for \( F \). Let \( V = \{ (x, y) : |x| \leq 1/\lambda \} \). There is some odd positive integer \( m \) such that \( (m - 1)\pi/2 \) and \( \theta_0 \) are between \( (m - 2)\pi/2 \) and \( m\pi/2 \).

Note that \( F(A_0) \cap V \) contains all points \( z \) of \( \tilde{A} \) such that \( z \) and \( F(z) \) are contained in \( A_0 \). Because each component of \( F(A_0) \cap V \) extends from one vertical boundary of \( V \) to the other, each component of \( J[F(A_0) \cap V] \) extends from one boundary of \( A_0 \) to the other, and there are exactly \( m \) components of this set. (We are applying Lemma 11 here.) Thus, each component of \( F[F(A_0) \cap V] \) intersects each component of \( [F(A_0) \cap V] \) and extends from one vertical boundary of each component of \( [F(A_0) \cap V] \) to the other. Hence, Lemma 11 guarantees that \( F[F(A_0) \cap V] \cap [F(A_0) \cap V] \) consists of exactly \( m^2 \) components, each extending from boundary to boundary and \( F[F(A_0) \cap V] \) consists of exactly \( m^2 \) components also, and the process can continue. At the \( n \)th iterate of \( F \), \( F^n(A_0) \cap A_0 \) consists of exactly \( m^{n-1} \) components, each extending from one boundary of \( A_0 \) to the other. Thus, it follows that \( \mathcal{C} = \bigcap_{n \geq 0} F^n(A_0) \) is a Cantor set of continua. Applying Lemma 11 and noting that if \( H \) is a vertical line and

\[
h_n = \max \{ ||\widetilde{H}|| : \widetilde{H} \text{ is a component of } H \cap (F^n(A_0) \cap A_0) \},
\]

then the size of \( \lambda \) forces \( \lim_{n \to \infty} h_n = 0 \), we also get that each component of \( \mathcal{C} \) is an arc and that \( \mathcal{C} = \{ x \in D_1(0) : F^n(x) \text{ is in } D_1(0) \text{ for all } n \} \) is an invariant Cantor set.
It follows from arguments quite similar to the standard horseshoe map arguments discussed in the second section that $F \mid C$ is conjugate to an $m$-shift and that $\tilde{A}$ is homeomorphic to the Knaster continuum $K_m$. □

Remember that the object we wish to study here is the boundary between the fluids. For $F$ a finite composition of the squish and stir maps, that means studying the following sets: let $B = \{x \in Y: \text{there is a sequence of increasing integers } n_1, n_2, \ldots \text{ and a sequence of points } x_1, x_2, \ldots \text{ of } \mathbb{R}_x \text{ such that } F^{n_1}(x_1), F^{n_2}(x_2), \ldots \text{ converges to } x\}$, and let $\tilde{B} = B \cup \{o: o \text{ is a bounded component of } Y \setminus B\}$. Now it is not difficult to prove that $B$ and $\tilde{B}$ are continua contained in $\tilde{A}$, but it would be nice if $\tilde{B}$ were the same set as $\tilde{A}$. The continuum $\tilde{B}$ is the continuum obtained when one has been alternating the application of the squish and stir maps for all time in the past, and considers the limit of the (augmented) $x$-axis $\mathbb{R}_x \cup \{\infty_+, \infty_-\}$. For a case in which a finite composition $F$ of the squish and stir maps is being considered (equivalently, a case in which an actual discrete dynamical system is under investigation, as opposed to an arbitrary sequence of applications of squishes and stirs), one can study the limit $\lim_{n \to \infty} F^{-n}(\mathbb{R}_x \cup \{\infty_+, \infty_-\})$ by studying $\lim_{n \to \infty} F^{n}(\mathbb{R}_x \cup \{\infty_+, \infty_-\})$. If this “limit” even exists (and we prove that for $F$ of the form $R_{\theta_0} J_\lambda$, it does), then it is an invariant continuum, and if we have found the latter, we have found the former. In one case $\tilde{B}$ is not $\tilde{A}$, and that is the case of Trivial Example 3, where $\tilde{A} = A$, but $B = \tilde{B} = D_1(0) \cup \mathbb{R}_x$. However, the next lemmas and Theorem 19 show that for compositions of the form $F = R J$, $\tilde{B} = \tilde{A}$.

From another point of view, studying the boundary between the red and blue fluids means looking at the boundary between the set of points that came from $\infty_+$ (the red points) and the set of points that came from $\infty_-$ (the blue points). Denote this set by $\mathcal{CF}$: precisely, let $G^+$ denote that component of $Y \setminus \tilde{A}$ that contains $\infty_+$, and let $G^-$ denote that component of $Y \setminus \tilde{A}$ that contains $\infty_-$. Then define $\mathcal{CF} = \{x \in Y: \text{for each open set } o \text{ that contains } x, o \text{ intersects at least two sets in the collection } \{(G^+)^o, (G^-)^o, (\tilde{B})^o\} \cup (\tilde{B})^o\}$. It is also the case that for $F = R J$, $\mathcal{CF} = \tilde{A}$. This is proved in Theorem 20. Essentially, proving that $\mathcal{CF} = \tilde{A}$ means proving that there are no “tendrils” or “hairs” of $\tilde{A}$ sticking above or below $\tilde{A}$ which are contained in $(Y \setminus \tilde{A})$.

Let $\theta_1 = \arctan(\lambda) - \arctan(1/\lambda)$.

**Lemma 14.**

(1) If $\theta_0 > \theta_1 = \arctan(\lambda) - \arctan(1/\lambda)$, then $F = R_{\theta_0} J_\lambda$ has a fixed point in quadrant I and a fixed point in quadrant III.

(2) If $\theta_0 \leq \theta_1$, then $F$ has only one fixed point (0) in $D_1(0)$.

(3) If $\theta_0 > \theta_1 + \pi$, then $F$ has period 2 points in each quadrant.

(4) If $\theta_0 > \pi/2$, then $F$ has period 4 points in each quadrant.

**Proof.** (1) Suppose that $\theta_0 > \theta_1 = \arctan(\lambda) - \arctan(1/\lambda)$. Suppose that $q$ denotes the unique point in $I \cap L_\lambda^+ \setminus S_{\theta_0}$. Then $J(q)$ is the unique point in $I \cap L_\lambda^{1/\lambda} \setminus S_{\theta_0}$, and $F(q) = RJ(q) = q$. There is likewise a fixed point in $\text{III} \cap L_\lambda^+ \setminus S_{\theta_0}$.
(2) Recall that each point of $M_2 \cup M_4$ moves closer to 0 under $F$, and each point of $M_1 \cup M_3$ moves further away from 0, while points on $L^+_\lambda \cup L^-_\lambda$ stay at the same distance from 0 under application of $F$. Since points on $L^-_\lambda$ move off $L^-_\lambda$ under $F$, as do points on $L^+_\lambda$ (because they do not get rotated back to $L^+_1$ after moving to $L^+_1$ under $J$), the result follows.

(3) Suppose that $\theta_0 > \theta_1 + \pi$. Suppose that $q_\pi$ denotes the unique point in $I \cap L^+_\lambda \cap S_{r_{\theta_1} + \pi}$. Then $F(q_\pi) = q'_\pi$ is the unique point in $I \cap L^+_\lambda \cap S_{r_{\theta_1} + \pi}$, and $F^2(q_\pi) = F(q'_\pi) = q_\pi$ is the unique point in $I \cap L^+_\lambda \cap S_{r_{\theta_1} + \pi}$. Thus, quadrants $I$ and $III$ contain the period 2 points $q_\pi$ and $q'_\pi$. Next suppose that $q_{\pi,II}$ denotes the unique point in $II \cap L^-_\lambda \cap S_{r_{\pi - \theta_1}}$. Then $F(q_{\pi,II}) = q'_{\pi,II}$ is the unique point in $IV \cap L^-_\lambda \cap S_{r_{\pi - \theta_1}}$, and, as before, there exist period 2 points in $III$ and $IV$.

(4) Suppose that $\theta_0 > \pi/2$. Then let $p_1$ denote the unique point in $I \cap L^-_\lambda \cap S_{r_{\pi/2}}$. Then $F(p_1) = p_2$ is the unique point in $II \cap L^-_\lambda \cap S_{r_{\pi/2}}$, $F^2(p_1) = p_3$ is the unique point in $III \cap L^+_\lambda \cap S_{r_{\pi/2}}$, and $F^3(p_1) = p_4$ is the unique point in $IV \cap L^-_\lambda \cap S_{r_{\pi/2}}$. Since $F^4(p_1) = p_1$, the result follows.

**Lemma 15.** Suppose $F = R\theta_0 J_\lambda$ for some $\lambda > 1$, $\theta_0 > \theta_1 = \arctan(\lambda) - \arctan(1/\lambda)$. If $x$ is in $M_1 \cup M_3$ and $x$ is outside $S_{r_{\theta_1}}$, then eventually $F^n(x)$ is not in $D_1(0)$. If $x$ is in $R(M_2^{-}) \cup R(M_4^{-})$ and $x$ is outside $S_{r_{\theta_1}}$, then eventually $F^{-n}(x)$ is not in $D_1(0)$.

**Proof.** Let $R = R\theta_0$ and $J = J_\lambda$. Without loss of generality, suppose $x$ is in $M_1$ and $x$ is not inside $S_{r_{\theta_1}}$. The proof for $x$ in $M_3$ is similar. Then $J(x)$ is in $M_1$, and $d(J(x), 0) > d(x, 0)$. Since $x$ is not in $S_{r_{\theta_1}}$, $J(x)$ is in $S_{r_{\theta}}$ for some $\theta < \theta_1$, and $R$ does not rotate $J(x)$ back to the ray containing $x$. Thus, $d(F(x), 0) > d(x, 0)$, and $F(x)$ is in $M_1$, but outside $S_{r_{\theta}}$. Then we can continue: it follows that $d(JF(x), 0) > d(F(x), 0) > d(x, 0)$, and applying $R$ next results in an even smaller rotation, so that $d(F^2(x), 0) > d(F(x), 0)$, and $F^2(x)$ is in $M_1$. In general, $d(F^n(x), 0) > d(F^{n-1}(x), 0)$, and $F^n(x)$ is in $M_1$. As in a previous argument, since the distances from 0 are strictly increasing, eventually the forward orbit of $x$ is outside $D_1(0)$, but remains in $M_1$.

The proof of the second part is similar and is omitted.

**Theorem 16.** Suppose $F = R\theta_0 J_\lambda$ for some $\lambda > 1$, and $0 < \theta_0 \leq \theta_1$. Then $\tilde{B} = \tilde{A} \simeq \mathbb{R}_x$. (See Fig. 9.)

**Proof.** It follows from Lemma 15 that each member of $M_1 \cup M_3$ is eventually mapped out of $D_1(0)$ by $F$, and that each member of $R(M_2^{-}) \cup R(M_4^{-})$ is eventually mapped out of $D_1(0)$ by $F^{-1}$. Further, 

$$(M_1 \cup M_3) \cup (R(M_2^{-}) \cup R(M_4^{-})) = \mathcal{Y},$$

and it is not difficult to see that there are four disjoint connected, open sets, $\Omega_1$, $\Omega_2$, $\Omega_3$, and $\Omega_4$ such that 

$$\Omega_1 = \{x \text{ in } \mathcal{Y} : x \text{ wanders from } \infty_y^+ \text{ to } \infty_x^+\},$$
\( \Omega_2 = \{ x \in Y : x \text{ wanders from } \infty^+_y \text{ to } \infty^-_x \} \),
\( \Omega_3 = \{ x \in Y : x \text{ wanders from } \infty^-_y \text{ to } \infty^-_x \} \), and
\( \Omega_4 = \{ x \in Y : x \text{ wanders from } \infty^-_y \text{ to } \infty^+_x \} \).

Further, we can apply Lemma 10 to conclude that, if \( L \) denotes the arc on \( L^+_\lambda \) from \( 0 \) to the boundary of \( D_1(0) \), then the Hausdorff limit \( L^+_{\infty_x} \) of the sequence \( L, F(L), F^2(L), \ldots \) is homeomorphic to a ray that begins at \( 0 \), has one-to-one projection \( \pi_x L^+_{\infty_x} \), and extends to \( \infty^+_x \). Further, under iteration by \( F \), each point of \( L^+_{\infty_x} \setminus \{0\} \) wanders from \( 0 \) to \( \infty^+_x \). Likewise there are sets \( L^-_{\infty_x}, L^+_{\infty_y}, \) and \( L^-_{\infty_y} \) each homeomorphic to a ray; each with initial point at \( 0 \); each having one-to-one projection, respectively; and each extending to \( \infty^-_x, \infty^+_y, \) or \( \infty^-_y \), respectively. Each point of \( L^-_{\infty_x} \) wanders from \( 0 \) to \( \infty^-_x \), each point of \( L^+_{\infty_y} \) wanders from \( \infty^+_y \) to \( 0 \), and each point of \( L^-_{\infty_y} \) wanders from \( \infty^-_y \) to \( 0 \). Then \( L^+_{\infty_x} \cup L^-_{\infty_x} \) is homeomorphic to \( \mathbb{R} \) and \( \mathbb{R}_x \), and the result follows, since

\[
Y \setminus \{\Omega_1, \Omega_2, \Omega_3, \Omega_4\} = L^+_{\infty_x} \cup L^-_{\infty_x} \cup L^+_{\infty_y} \cup L^-_{\infty_y}.
\]

Theorem 17. Suppose \( F = R_{\theta_0} J_\lambda \) for some \( \lambda > 1 \), and \( \pi/2 < \theta_0 \). Suppose that \( q \) denotes the unique point in \( I \cap L^+_\lambda \cap S_{r_q} \), and \( p_1 \) denotes the unique point on \( I \cap L^+_\lambda \cap S_{r_{p_1}} \). Then \( q \) is a fixed point for \( F \) and \( p_1 \) is a period 4 point for \( F \). Let \( P \) denote the unique arc on \( \mathbb{R} \cap L^+_\lambda \) that extends from \( p_1 \) to \( q \). For each integer \( n \), there exists a unique subinterval \( P_n \) of the arc \( P \) which has initial point \( t_n \) and terminal point \( q \) on \( S_{r_q} \) such that

1. \( P_{n+1} \subseteq P_n \),
2. \( F^n(P_n) \) is the graph of some continuous map \( f_n : [a_n, b_n] \rightarrow [-1, 1] \) such that
   a. \( [a_n, b_n] \subseteq [-1, 1] \) and \( \text{gr}(f_n([a_n, b_n])) \subseteq \mathbb{M}_2 \), where
   \[
   \text{gr}(f_n([a_n, b_n])) = \{ (x, f_n(x)) : x \text{ is in } [a_n, b_n] \},
   \]
   b. if \( x < y \) in \( [a_n, b_n] \), then \( 0 < f_n(x) < f_n(y) \), and
   c. if \( x < y \) in \( [a_n, b_n] \), then \( ||(x, f_n(x))|| < ||(y, f_n(y))|| \), and
3. if \( \Theta \) is a ray contained in \( \overline{\mathbb{M}_2} \), then \( \Theta \) intersects \( F^n(P_n) \) in the unique point \( q_{n\Theta} \), and
   \[
   ||q_1\Theta|| < ||q_2\Theta|| < \cdots < ||q_{n\Theta}|| < \cdots < 1.
   \]
The arcs \( F^1(P_1), F^2(P_2), \ldots \) converge (with respect to the Hausdorff metric) to the arc \( Q \), which

1. is contained in \( \overline{\mathbb{M}_2} \cap \overline{D_{r_q}(0)} \),
2. is the graph of some nondecreasing function \( f : [a, b] \rightarrow [-1, 1] \), with \( [a, b] \subseteq [-1, 1] \),
3. has the property that if \( x < y \) in \( [a, b] \), then \( ||(x, f(x))|| \leq ||(y, f(y))|| \), and
4. has the property that if \( \Theta \) is a ray contained in \( \overline{\mathbb{M}_2} \), then \( \Theta \) intersects \( Q \) in the unique point \( q_\Theta \), and \( q_{1\Theta}, q_{2\Theta}, q_{3\Theta}, \ldots \) converges to the point \( q_\Theta \).

Moreover, \( F(Q) \supseteq Q \) and \( Q \subseteq A \).

Proof. Let \( P \) denote the arc on \( L^+_\lambda \) which has initial point \( p_1 \) on \( S_{r_{p_1/2}} \) and terminal point \( q \) on \( S_{r_q} \), and let \( P_1 = P \) and \( t_1 = p_1 \). Then \( F(P) \) satisfies Properties (2) and (3) of the
statement. (Note that $J(P)$ is an arc on $L^{+}_{1/\lambda}$ which has initial point $J(t_1)$ on $S_{\pi/2}$ and terminal point $J(q)$ on $S_{\pi/2}$. It follows from Lemma 10 that $F(P)$ has the properties desired.) Also, $F(P) \cup P$ is an arc, and $F(P)$ intersects each $S_{r}$ for $\pi/2 \leq t \leq \theta_1$ in exactly one point, as does $P$. Moreover, $F(P) \cap S_{r}$ intersects a ray lying to the left of the ray on which $P \cap S_{r}$ lies except that $F(P) \cap S_{\pi/2} = P \cap S_{\pi/2} = \{q\}$, and $F(P)$ lies in $\overline{M}_{2}$.

Then $F^2(P) \cup F(P)$ is also an arc. Since Lemma 10 can be applied to $F(P) \cap I$ and $JF(P) \cap I$ must intersect $S_{\pi/2}$, it follows that

$$F^2(P) \cap \overline{M}_{2} \cap (\overline{D}_{r_{q_1}}(0) \setminus D_{\pi/2}(0)),$$

is an arc which extends from some unique point $q_{2\alpha}$ on the ray $\Theta = \alpha$ (which is actually that portion of $L^-_{\alpha}$ above the $x$-axis), where $\alpha = \pi - \arctan \lambda$, to the point $q$. Let

$$P_2 = F^{-2}(F^2(P) \cap \overline{M}_{2} \cap (\overline{D}_{r_{q_1}}(0) \setminus D_{\pi/2}(0))),$$

so that $P_2$ is an arc on $P = P_1$. Then $P_2$ satisfies Properties (1)–(3) of the statement, and also $F^2(P_2)$ intersects each $S_{r}$ for $\pi/2 \leq t \leq \theta_1$ in at most one point, as does $F(P_1)$. Moreover, $F^2(P_2) \cap S_{r}$ intersects a ray lying to the left of the ray on which $F(P_1) \cap S_{r}$ lies except that $F^2(P_2) \cap S_{\pi/2} = F(P_1) \cap S_{\pi/2} = \{q\}$, and $F^2(P_2)$ lies in $\overline{M}_{2}$.

We can continue this process: at the $n$th level, $F^n(P_{n-1}) \cup F^{n-1}(P_{n-1})$ is an arc. Lemma 10 can be applied to $F^{n-1}(P_{n-1}) \cap I$ and $JF^{n-1}(P_{n-1}) \cap I$ while $JF^{n-1}(P_{n-1}) \cap I$ must intersect $S_{\pi/2}$, so that

$$F^n(P) \cap \overline{M}_{2} \cap (\overline{D}_{r_{q_1}}(0) \setminus D_{\pi/2}(0)),$$

is an arc which extends from some unique point $q_{n\alpha}$ on the ray $\Theta = \alpha$, where $\alpha = \pi - \arctan \lambda$, to the point $q$. Let

$$P_n = F^{-n}(F^{n-1}(P_{n-1}) \cap \overline{M}_{2} \cap (\overline{D}_{r_{q_1}}(0) \setminus D_{\pi/2}(0))),$$

so that $P_n$ is an arc contained in $P = P_{n-1}$. Then $P_n$ satisfies Properties (1)–(3) of the statement, and also $F^n(P_n)$ intersects each $S_{r}$ for $\pi/2 \leq t \leq \theta_1$ in exactly one point, as does $F^{n-1}(P_{n-1})$. Moreover, $F^n(P_n) \cap S_{r}$ intersects a ray lying to the left of the ray on which $F^{n-1}(P_{n-1}) \cap S_{r}$ lies except that $F^n(P_n) \cap S_{\pi/2} = F^{n-1}(P_{n-1}) \cap S_{\pi/2} = \{q\}$, and $F^n(P_n)$ lies in $\overline{M}_{2}$. Thus, the first part of the statement has been proved.

It now follows from the fact that each $P_n$ satisfies Properties (2) and (3), that the Hausdorff limit of the arcs $F^1(P_1), F^2(P_2), \ldots$ is an arc $Q$, which

1. is contained in $\overline{M}_{2} \cap \overline{D}_{r_{q_1}}(0)$,
2. is the graph of some nondecreasing function $f : [a, b] \to [-1, 1]$, with $[a, b] \subseteq [-1, 1]$,
3. has the property that if $x < y$ in $[a, b]$, then $\|x, f_n(x)\| \leq \|y, f_n(y)\|$, and
4. has the property that if $\Theta$ is a ray contained in $\overline{M}_{2}$, then $\Theta$ intersects $Q$ in the unique point $q_{\Theta}$, and $q_{\Theta}, q_{2\Theta}, q_{3\Theta}, \ldots$ converges to the point $q_{\Theta}$. Moreover, $F(Q) \supseteq Q$ and $Q \subseteq \overline{A}$. □

**Theorem 18.** Suppose that $F = R_{0\lambda}J_{\lambda}$ for some $\lambda > 1$ and $\theta_0 > \theta_1$. Suppose that $q$ denotes the unique point in $I \cap L^{+}_{\lambda} \cap S_{r_{q_1}}$, if $\theta_0 > \theta_1$; $q = 0$, otherwise. Then $q$ is a
There exists an arc $P^\infty$ on the line $L_\lambda^+$ which has initial point $q$ and terminal point $q_1$ on $S_1$ such that

1. $F^n(P^\infty)$ is the graph of some continuous map $f_n : [a, b_n] \to [-1, 1]$ such that
   a. $[a, b_n] \subseteq [0, \infty)\times[a, b_n]$ where $\pi_x q$ and $\text{gr}(f_n([a, b_n])) \subseteq \overline{M}_1$, and
   b. if $x < y$ in $[a, b_n]$, then $\|f_n(x)\| < \|f_n(y)\|$, and
2. if $\Theta$ is a ray contained in $\overline{M}_1$, then $\Theta$ intersects $F^n(P^\infty)$ in the unique point $q^\infty_\Theta$ and the sequence $\|q_1\|, \|q_2\|, \|q_3\|, \ldots$ is decreasing to some finite value.

The arcs $F^1(P^\infty), F^2(P^\infty), \ldots$ converge (with respect to the Hausdorff metric) to the arc $Q^\infty$, which

1. is contained in $\overline{M}_1$,
2. is the graph of some continuous function $f : [a, \infty^+] \to [\pi y q, \infty^+]$,
3. has the property that if $x < y$ in $[a, \infty^+]$, then $\|f(x)\| \leq \|f(y)\|$, and
4. has the property that if $\Theta$ is a ray contained in $\overline{M}_1$, then $\Theta$ intersects $Q^\infty$ in the unique point $q^\infty_\Theta$ and $q^\infty_\Theta, q^\infty_\Theta, q^\infty_\Theta, \ldots$ converges to the point $q^\infty_\Theta$.

Moreover, $F(Q^\infty) = Q^\infty$ and $Q^\infty \subseteq A$.

Because of symmetry, there must likewise exist an arc $Q^-\infty$, and a fixed point $q_-$ in $\Pi$ (which is the unique point in $\Pi \cap L_\lambda^+ \cap S_{\theta_0}$), with the properties that

1. $Q^-\infty$ is contained in $\overline{M}_3$,
2. $Q^-\infty$ is the graph of some continuous function $f : [\infty^-, a_-] \to [\pi y q, \infty^+]$ (where $\pi_x q_- = a$),
3. if $x < y$ in $[\infty^-, a_-]$, then $\|f(x)\| \geq \|f(y)\|$, and
4. if $\Theta$ is a ray contained in $\overline{M}_3$, then $\Theta$ intersects $Q^-\infty$ in the unique point $q^-\infty_\Theta$.

Moreover, $F(Q^-\infty) = Q^-\infty$ and $Q^-\infty \subseteq \tilde{A}$.

**Proof.** This proof is similar to the last, but is actually easier, so we omit it. □

**Theorem 19.** If $F = R_{\theta_0} J_\lambda$ for some $\lambda > 1$, and $\theta_0 > 0$, then $\tilde{B} = \tilde{\tilde{A}}$. Further, $\tilde{B}$ and $B$ are continua in $Y$ that separate $Y$, $Y \setminus \tilde{B}$ has exactly two components, and $F(B) = B$, $F(\tilde{B}) = \tilde{B}$.

**Proof.** Clearly, $B \subseteq \tilde{B} \subseteq \tilde{A}$. Thus, we need to show that $\tilde{B} \supseteq \tilde{\tilde{A}}$. Since for each $n$, $F^n(\mathbb{R}_x)$ is a continuum contained in $F^n(A)$, any Hausdorff limit of a subsequence of $F^1(\mathbb{R}_x), F^2(\mathbb{R}_x), \ldots$ is a continuum contained in $\tilde{A}$. Since each $F^n(\mathbb{R}_x)$ contains $\{0, \infty^+, \infty^\circ\}$, the union of all Hausdorff limits of these subsequences is a continuum contained in $A$. It follows from the previous lemmas that $\tilde{B} \setminus \tilde{B} \subseteq L_1(0)$, $F'(B) = B$, and $F(\tilde{B}) = \tilde{\tilde{B}}$. Since each $F^n(\mathbb{R}_x)$ separates $Y$, so do $\tilde{B}$ and $B$. Also, it should be clear that $Y \setminus \tilde{B}$ has exactly two unbounded components. Let $U^+$ denote that unbounded component of $Y \setminus \tilde{B}$ that contains $\infty^+_y$, and $U^-$ denote that unbounded component of $Y \setminus \tilde{B}$ that contains $\infty^-_y$. Then $Y \setminus \tilde{B} = U^+ \cup U^-$, $F(U^+) = U^+$, and $F(U^-) = U^-$. We write $RJ$ for $R_{\theta_0} J_\lambda$. If $\theta_0 \leq \theta_1$, then we can apply Theorem 16 to get the result. Hence, we assume that $\theta_0 > \theta_1$. If $x$ is in $\tilde{A} \setminus \tilde{B}$ then without loss of generality,
Fig. 15. The figure shows some of the sets involved in the proofs of Theorems 19 and 20.

we can assume $x$ is in $D_1(0)$: if $x$ is not in $D_1(0)$, then there exists some integer $N$ such that if $n \geq N$, then $F^{-n}(x)$ is in $D_1(0)$. Note also that $F(\tilde{A}\setminus \tilde{B}) = \tilde{A}\setminus \tilde{B}$, and $(\tilde{A}\setminus \tilde{B}) \cap D_1(0) \neq \emptyset$. See Fig. 15. Further, we may assume that $x$ is not the fixed point $q$ of Lemma 14(1), which is on $I \cap L^+_{1/\lambda} \cap S_{r_{\theta}}$, since $\tilde{A}\setminus \tilde{B}$ must be uncountable.

Claim (See previous lemmas).

1. If $x$ is in $R(M_1^- \cup M_\omega^-)$, then $d(F^{-1}(x), 0) > d(x, 0)$.
2. If $x$ is in $R(M_1^+ \cup M_\omega^+)$, then $d(F^{-1}(x), 0) < d(x, 0)$.

Suppose then that $z \neq q$ is in $\tilde{A}\setminus \tilde{B}$, $(I \cup II) \cap D_1(0)$. There is some arc $T_z$ such that $z$ is one endpoint of $T_z$, $\infty^+_y$ is the other endpoint of $T_z$, but $T_z \cap \tilde{B}$ is empty and $q$ is not in $T_z$. If for infinitely many $n$, $F^{-n}(T_z) \cap \mathbb{R}_x$ is not empty, then for each of those $n$, there exists some $z_n$ in $F^{-n}(T_z) \cap \mathbb{R}_x$, and $F^n(z_n)$ in $T_z \cap F^n(\mathbb{R}_x)$. Since $T_z$ is compact, we may assume that some sequence $F^{n_1}(z_{n_1}), F^{n_2}(z_{n_2}), \ldots$ converges to the point $z'$ in $T_z$, which implies that $T_z \cap \tilde{B}$ is not empty. Thus, there is some positive integer $N_1$ such that if $n \geq N_1$, then $F^{-n}(T_z) \cap \mathbb{R}_x = \emptyset$. It also follows that for each $n \geq N_1$, $F^{-n}(T_z) \cap (III \cup IV) = \emptyset$, for otherwise $F^{-n}(T_z) \cap \mathbb{R}_x \neq \emptyset$. We can assume then without loss of generality that for each $n$, $F^{-n}(z) \notin (III \cup IV \cup \mathbb{R}_x)$ and $F^{-n}(T_z) \cap \mathbb{R}_x = \emptyset$.

Note that any point $z \neq q$ in $\tilde{A}\setminus \tilde{B}$, $(I \cup II)$ is eventually in $I$ under iteration by $F^{-1}$: such a $z$ cannot be undergoing rotation by $R^{-1}$ that is enough to rotate it across the $x$-axis (and all the way back into $II$), for then some point of $T_z$ is mapped across the $x$ axis, and that can’t happen. Thus, each application of $R^{-1}$ moves $z$ closer to the positive $y$-axis in $II$ or into $I$, and then $J^{-1}$ moves it closer still to the positive $y$-axis, but moves no point from $I$ into $II$ or $III$ or $IV$. Thus, if $z$ is in $II$, then eventually $F^{-n}(z)$ gets moved over the positive $y$-axis, never to return. Thus, we can assume that each such $z$ is in $\tilde{A}\setminus \tilde{B} \cap I$, and $T_z$ does not contain $q$.

Now if for some $n$, $F^{-n}(z)$ is in $D_{r_{\theta}}(0) \cap R(M_1^+)$. 


then for all \( m > n \), \( F^{-m}(z) \) is in \( D_{r_{q_1}}(0) \cap R(M_1^-) \): for such a point \( \bar{x} = F^{-n}(z) \), \( R^{-1}(\bar{x}) \) is rotated by some \( \theta \geq \theta_1 \) in the clockwise direction, but remains the same distance from 0. Then \( J^{-1}R^{-1}(\bar{x}) = F^{-1}(\bar{x}) \) is moved closer to 0 if \( R^{-1}(\bar{x}) \) is in \( M_1^- \), and is at the same distance from 0 if \( R^{-1}(\bar{x}) \) is in \( M_1^- \setminus M_i^- \) and \( F^{-1}(\bar{x}) \) is in \( M_i^- \). Then \( R^{-1}F^{-1}(\bar{x}) \) is rotated by at least \( \theta_1 \) in the clockwise direction, but not across the \( x \)-axis, so that \( R^{-1}F^{-1}(\bar{x}) \) is in \( M_1^- \), and again \( J^{-1}R^{-1}F^{-1}(\bar{x}) = F^{-2}(\bar{x}) \) is moved closer to the origin. This process continues, with the clockwise rotation always greater than \( \theta_1 \). But then eventually, \( F^{-m}(z) \) must get into \( M_1^- \), and stay there, from which it follows that it must eventually cross the \( x \)-axis. This cannot happen. Thus, for each \( n \), \( F^{-n}(z) \) is not in

\[
\overline{D_{r_1}(0) \cap R(M_1^-)}.
\]

Suppose \( F^{-n}(z) \neq q \) is in

\[
\overline{D_1(0) \setminus D_{r_{q_1}}(0) \cap R(M_2^-)}
\]

for some \( n \). It follows from Lemma 15 that for some \( m > n \), \( F^{-m}(z) \) is in \( M_2^- \setminus D_1(0) \) and it can never return, which is a contradiction to \( z \) being an element of \( \tilde{A} \). Likewise, \( F^{-m}(z) \) cannot be a member of

\[
\{ y \in \overline{D_1(0) \cap R(M_1^+)} : y \text{ is above } Q^\infty \}.
\]

Let

\[
U_1 = \{ y \in \overline{D_1(0) \cap R(M_1^-)} : y \text{ is above } Q^\infty \} \cup \overline{D_{r_{q_1}}(0) \cap R(M_1^-)}.
\]

Then the backward orbit of \( z \) cannot enter \( U_1 \), and for each \( n \), \( F^{-n}(z) \) is in

\[
(D_{r_{q_1}}(0) \cap R(M_2^-)) \cup (\overline{D_1(0) \setminus D_{r_{q_1}}(0)} \cap \overline{R(M_1^- \setminus U_1)}).
\]

Now it cannot be the case that for each \( n \), \( F^{-n}(z) \) is in \( (D_{r_{q_1}}(0) \cap R(M_2^-)) \), for then eventually the backward orbit moves out of the unit disk and out of \( \tilde{A} \). Likewise, if for each \( n \), \( F^{-n}(z) \) is in \( (\overline{D_1(0) \cap R(M_1^-)}), \) then eventually the backward orbit of \( z \) moves across the \( x \)-axis. Thus, for infinitely many \( n \), \( F^{-n}(z) \) is in \( (D_{r_{q_1}}(0) \cap R(M_2^-)) \) and, for infinitely many \( n \), \( F^{-n}(z) \) is in

\[
((\overline{D_1(0) \setminus D_{r_{q_1}}(0)}) \cap R(M_1^-)) \setminus U_1.
\]

Choose \( n_1 \) such that \( F^{-(n_1-1)}(z) \) is in \( D_{r_{q_1}}(0) \cap R(M_2^-) \), and \( F^{-(n_1-1)}(z) \) is in

\[
(D_1(0) \setminus D_{r_{q_1}}(0)) \cap R(M_1^-).
\]

Let \( z' = F^{-n_1}(z) \). But then \( R^{-1}(z') \) is in \( D_{r_{q_1}}(0) \) and \( J^{-1} \) moves \( R^{-1}(z') \) closer to the positive \( y \)-axis, which means that \( F^{-1}(z') \) is not in \( (\overline{D_1(0) \setminus D_{r_{q_1}}(0)}) \cap R(M_1^-) \). Again we have a contradiction.

Then no point above \( \tilde{B} \) is in \( \tilde{A} \setminus \tilde{B} \). By symmetry, no point below \( \tilde{B} \) is in \( \tilde{A} \). Thus, \( \tilde{A} = \tilde{B} \). \( \Box \)
Theorem 20. If $F = R_{\theta_0} J_{\lambda}$ for some $\lambda > 1$ and $\theta_0 > 0$, then $\mathcal{CF} = \tilde{B} = \tilde{A}$.

Proof. Clearly, $\mathcal{CF} \subseteq \tilde{A}$. Thus, we need to show that $\mathcal{CF} \supseteq \tilde{A}$. Now $F(\mathcal{CF}) = \mathcal{CF}$, and

$$Y \setminus \mathcal{CF} = (V^+) \cup (V^-),$$

with $F(V^+) = (V^+)$, and $F(V^-) = (V^-)$. We show that $(V^+) = V^+$. The other case follows by symmetry.

We write $RJ$ for $R_{\theta_0} J_{\lambda}$. If $x$ is in $(\tilde{A} \setminus \mathcal{CF}) \cap ((V^+) \setminus V^+)$, then without loss of generality, we can assume $x$ is in $D_1(0)$; if $x$ is not in $D_1(0)$, then there exists some integer $N$ such that if $n \geq N$, then $F^{-n}(x)$ is in $D_1(0)$. Note also that $F(\tilde{A} \setminus \mathcal{CF}) = \tilde{A} \setminus \mathcal{CF}$, and $(\tilde{A} \setminus \mathcal{CF}) \cap D_1(0) \neq \emptyset$.

Claim (See previous lemmas).

1. If $x$ is in $R(M_2^- \cup M_4^-)$, then $d(F^{-1}(x), 0) > d(x, 0)$.
2. If $x$ is in $R(M_1^- \cup M_3^-)$, then $d(F^{-1}(x), 0) < d(x, 0)$.

Suppose then that $z \neq q$ is in $(\tilde{A} \setminus \mathcal{CF}) \cap (V^+)^c \cap D_1(0)$. There is some arc $T_z$ such that $z$ is one endpoint of $T_z$, $\infty^+_y$ is the other endpoint of $T_z$, but $T_z \cap \mathcal{CF} = \emptyset$ and $q \notin T_z$. If for some $n$, $F^{-n}(T_z) \cap \mathcal{CF} \neq \emptyset$, then there exists some $z_n$ in $F^{-n}(T_z) \cap \mathcal{CF}$, and $F^n(z_n) \in T_z \cap F^n(\mathcal{CF}) \neq \emptyset$. Thus, for each $n$, $F^{-n}(T_z) \cap \mathcal{CF} = \emptyset$, and it follows that $F^{-n}(T_z) \cap (V^-)^c = \emptyset$.

Note that any point $x$ in $\Pi \cap (\tilde{A} \setminus \mathcal{CF}) \cap (V^+)^c$ is eventually in $I$ under iteration by $F^{-1}$. (First, no point can map under $R^{-1}$ from $\Pi$ into $\Pi$ or back to $\Pi$ after crossing the $x$-axis. Doing so would map some point of $T_z$ to $\mathcal{CF}$ under $F^{-1}$.) However, as in the previous argument, no such point $x$ can stay in $\Pi$ or $IV$ under iteration by $F^{-1}$. Thus, eventually, $F^{-n}(x)$ gets moved over the positive $y$-axis, never to return. Likewise, if $x$ is in $IV \cap (\tilde{A} \setminus \mathcal{CF}) \cap (V^+)^c$, then eventually $F^{-n}(x)$ is moved across the negative $y$-axis, never to return. But this would force some image of $T_z$ to intersect $\mathcal{CF}$, and that cannot happen. Hence, we can assume that $x$ is in $(I \cup \Pi) \cap (\tilde{A} \setminus \mathcal{CF}) \cap (V^+)^c$. As before, we assume $x \neq q$.

Now if for some $n$, $F^{-n}(x)$ is in

$$\overline{D_{r_0}(0) \cap R(M_1^-)},$$

then for all $m > n$, $F^{-m}(x)$ is in $D_{r_0}(0) \cap R(M_1^-)$. For such a point $\tilde{x} = F^{-n}(x)$, $R^{-1}(\tilde{x})$ is rotated by some $\theta \geq \theta_1$ in the clockwise direction, but remains the same distance from 0. Then $J^{-1} R^{-1}(\tilde{x}) = F^{-1} (\tilde{x})$ is moved closer to 0 (or if $R^{-1}(\tilde{x})$ is in $M_1^- \setminus M_1^-$, same distance to 0), since $F^{-1}(\tilde{x})$ must be in $R(M_1^-)$. But then $R^{-1} F^{-1} (\tilde{x})$ is rotated by more than $\theta_1$ in the clockwise direction, and again $J^{-1} R^{-1} F^{-1}(\tilde{x}) = F^{-2}(\tilde{x})$ is moved closer to the origin. This process continues, with the clockwise rotation always greater than $\theta_1$. But then eventually, $F^{-m}(x)$ must cross the $x$-axis, enter $IV$ and intersect $\mathcal{CF}$, since eventually it is mapped out of $IV$ by $F^{-1}$, and this cannot happen. Thus, for each $n$, $F^{-n}(x)$ is not in $\overline{D_{r_0}(0) \cap R(M_1^-)}$, and, similarly, $F^{-n}(x)$ is not in $\overline{D_{r_0}(0) \cap R(M_3^-)}$. 


Suppose \( F^{-n}(x) \neq q \) is in
\[
(D_1(0) \setminus D_{r_0}(0)) \cap R(M_2^-)
\]
for some \( n \). It follows from Lemma 15 that for some \( m > n \), \( F^{-m}(x) \) is in \( M_2^- \setminus D_1(0) \) and it can never return, which is a contradiction to \( x \) being an element of \( \tilde{A} \). Likewise, \( F^{-m}(x) \) cannot be a member of \( \{ y \in R(M_1^-) \cap D_1(0) : y \text{ is above } Q^\infty \} \). Let
\[
U_I = \{ y \in R(M_1^-) \cap D_1(0) : y \text{ is above } Q^\infty \} \cup (R(M_1^-) \cap D_{r_0}(0)),
\]
and let
\[
U_III = \{ y \in R(M_1^-) \cap D_1(0) : y \text{ is above } Q^\infty \}.
\]
Then the backward orbit of \( x \) cannot enter \( U_I \cup U_III \), and for each \( n \), \( F^{-n}(x) \) is in
\[
(R(M_2^-) \cap D_{r_0}(0)) \cup ((D_1(0) \setminus D_{r_0}(0)) \cap R(M_1^-))
\]
or for each \( n \), \( F^{-n}(x) \) is in
\[
(R(M_2^-) \cap D_{r_0}(0)) \cup ((D_1(0) \setminus D_{r_0}(0)) \cap R(M_3^-)).
\]
Suppose that for each \( n \), \( F^{-n}(x) \) is in
\[
(R(M_2^-) \cap D_{r_0}(0)) \cup ((D_1(0) \setminus D_{r_0}(0)) \cap R(M_1^-)).
\]
The other case is similar. As before, it cannot be that for each \( n \), \( F^{-n}(x) \) is in \( (R(M_2^-) \cap D_{r_0}(0)) \), for then eventually the backward orbit moves out of the unit disk and out of \( \tilde{A} \). Likewise, if for each \( n \), \( F^{-n}(x) \) is in \( (R(M_1^-) \cap D_1(0)) \), then eventually the backward orbit of \( x \) moves across the \( x \)-axis, into \( IV \), and across \( CF \). Thus, for infinitely many \( n \), \( F^{-n}(x) \) is in \( (R(M_2^-) \cap D_{r_0}(0)) \), and for infinitely many \( n \), \( F^{-n}(x) \) is in \( (D_1(0) \setminus D_{r_0}(0)) \cap R(M_1^-) \). Choose \( n_1 \) such that \( F^{-n_1}(x) \) is in \( D_{r_0}(0) \cap R(M_2^-) \), and \( F^{-n_1-1}(x) \) is in \( (D_1(0) \setminus D_{r_0}(0)) \cap R(M_1^-) \). Let \( F^{-m_1}(x) = z' \). Then \( R^{-1}(z') \) is in \( M_2^- \), and \( F^{-1}(z') = J^{-1}R^{-1}(z') \) is further away from 0 than is \( R^{-1}(z') \) or \( z' \), and \( F^{-1}(z') \) cannot be in \( (D_1(0) \setminus D_{r_0}(0)) \cap R(M_1^-) \), so again we have a contradiction. Likewise the analogous situation in \( III \) cannot occur.

Hence, no point is in \( (V^+)^o \setminus V^+ \). By symmetry, no point is in \( (V^-)^o \setminus V^- \). Thus \( \tilde{A} = CF \), \( (V^+)^o = V^+ \) and \( (V^-)^o = V^- \). \( \square \)

**Lemma 21.** If \( \epsilon > 0 \), \( F \) is a finite composition of \( R_{\theta_0} \)'s and \( J_\lambda \)'s (for some \( \lambda > 1 \) and \( \theta_0 > 0 \)), and \( D_e(0) \) is not contained in \( \tilde{A} \), then there is a one-to-one image \( E \) of a ray which extends from the initial point \( \infty^+_y \) and intersects \( D_e(0) \). Also, \( F(E) = E \), \( E \cap \tilde{A} = \emptyset \), and a dense open subset \( E' \) of \( E \) has the property that if \( z \) is in \( E' \), then \( z \) wanders from \( \infty^+_y \) to either \( \infty^+_z \) or \( \infty^-_z \).

**Proof.** Since \( D_e(0) \setminus \tilde{A} \neq \emptyset \), we can choose some point \( x \) in \( D_e(0) \setminus \tilde{A} \). Note that \( D_e(0) \setminus \tilde{A} \) is an open set, and for some least \( n \), \( F^{-n}(x) \) is not in \( A_0 \). There is some open set \( u \) such that \( x \) is in \( u \subseteq D_e(0) \setminus \tilde{A} \), and \( F^{-n}(u) \subseteq A_{n-1} \setminus A_0 \). Then \( \{ F^{-m}(u) : m \geq n \} \) is mutually disjoint, which in turn implies that \( \{ F^{-m}(x) \}_{m \geq 0} \) converges to \( \infty^+_y \), and
\[ \langle F^m(x) \rangle_{m \geq 0} \text{ converges to either } \infty^+_x \text{ or } \infty^-_x. \] Since \( F^{-n}(x) \) is in \( A_{-1} \setminus A_0 \), there is an arc \( P_x \) from \( F^{-n}(x) \) to \( F^{-n-1}(x) \) in \( (A_{-2} \cup A_{-1}) \setminus A_0 \) with \( F^{-1}(P_x) \cap P_x = \{ F^{-n-1}(x) \} \). Let \( E = \{ \infty^+_y \} \cup \bigcup_{m \in \mathbb{Z}} F^m(P_x) \). Then \( E \) is the set with the properties desired. \( \Box \)

4. Results for random sequences of maps

Suppose that for given \( \lambda > 1 \) and \( \theta_0 > 0 \), \( \Sigma_{\theta_0, \lambda} \) is the set of sequences \( e = (e_1, e_2, \ldots) \) with the property that for each \( n \) in \( \mathbb{N} \), \( e_{-n} = \lambda \) or \( e_{-n} = \theta_0 \). For each \( e = (e_1, e_2, \ldots) \) in \( \Sigma_{\theta_0, \lambda} \), let

\[ W_e = \bigcap_{m \geq 1} e_{-1} \circ e_{-2} \circ \cdots \circ e_{-m}(A). \]

Since \( R_{\theta_0}(A) \subseteq A \) and \( J_\lambda(A) \subseteq A \), for each such sequence \( e = (e_1, e_2, \ldots) \),

\[ A \supseteq e_{-1}(A) \supseteq e_{-1} \circ e_{-2}(A) \supseteq \cdots \supseteq W_e. \]

In fact, the Hausdorff limit of the sequence \( A, e_{-1}(A), e_{-1} \circ e_{-2}(A), \ldots \) is the continuum \( W_e \). Let \( W_{\theta_0, \lambda} = \{ W_e : e \text{ is in } \Sigma_{\theta_0, \lambda} \} \). Now \( W_{\theta_0, \lambda} \) is a subset of the space \( \mathcal{F}(A) \) of all continua contained in \( A \), when that space of continua is endowed with the topology induced by the Hausdorff metric on \( \mathcal{F}(A) \). That topology is the same as the Vietoris topology on \( \mathcal{F}(A) \), and \( \mathcal{F}(A) \) is a compact metric space. If \( \{ b_1, b_2, \ldots, b_k \} \) is a finite collection of open sets in \( A \), then \( [b_1, b_2, \ldots, b_k] \) denotes the set of all points (continua) \( K \in \mathcal{F}(A) \) such that \( K \subseteq \bigcup_{i=1}^k b_i \) and \( K \cap b_i \neq \emptyset \) for any \( i \in \{1, \ldots, k\} \). The collection

\[ \{ [b_1, b_2, \ldots, b_k] : \{ b_1, b_2, \ldots, b_k \} \text{ is a finite collection of open sets in } A \} \]

is a basis for the Vietoris topology on \( \mathcal{F}(A) \).

Define a block to be a finite sequence \( d = \langle d_1, d_2, \ldots, d_n \rangle \) with the property that for each \( m \) such that \( 1 \leq m \leq n \), \( d_m = \theta_0 \) or \( d_m = \lambda \). We are particularly interested in sequences in \( \Sigma_{\theta_0, \lambda} \) that contain certain blocks, and usually those that contain an infinite collection of those blocks. To be precise, we say that the sequence \( e = (e_{-1}, e_{-2}, \ldots) \) contains the block \( d = \langle d_1, d_2, \ldots, d_n \rangle \) if, for some \( k \), \( e_{-k-m} = d_m \) for \( 1 \leq m \leq n \).

We say that the sequence \( e = (e_{-1}, e_{-2}, \ldots) \) is periodic if there exists some block \( d = \langle d_1, d_2, \ldots, d_n \rangle \) such that \( e \) consists of the block \( d \) repeated over and over again, i.e., for each positive integer \( j \), \( e_{-i-nj} = d_i \) for \( 1 \leq i \leq n \). The collection \( W_{\theta_0, \lambda}^Q = \{ W_e \in W_{\theta_0, \lambda} : W_e \text{ is periodic} \} \) is countable, because, of course, the collection of all blocks is countable.

The standard measure on \( \Sigma_{\theta_0, \lambda} \) is the product measure \( \mu_\infty \), where in the discrete factor space \( \{ R_{\theta_0}, J_\lambda \} \), the measure \( \mu(R_{\theta_0}) = 1/2 = \mu(J_\lambda) \). It is known that the measure of the collection \( \Sigma_{\theta_0, \lambda} = \{ e \in \Sigma_{\theta_0, \lambda} : e \text{ contains an infinite collection of copies of every block} \} \) has measure 1, i.e., \( \mu_\infty(\Sigma_{\theta_0, \lambda}) = 1 \). This fact is used in the proof of our last theorem. Suppose that \( \mu_A \) denotes a measure on \( A \) such that \( \mu_A(C) < \infty \) for each compact
subset $C$ of $A$ that does not contain either $\infty_+^x$ or $\infty_-^x$, and $\mu_A(o) = \mu_A(R_{\theta_0}(o)) = \mu_A(J_\lambda(o)) > 0$ for $o$ a nonempty, open set in $A$.

**Proposition 22.** Suppose $\lambda > 1$ and $\theta_0 > 0$. Then $W = W_{\theta_0, \lambda}$ is homeomorphic to a Cantor set, when $W$ is topologized by the Hausdorff metric.

**Proof.** Note that the collection $\widetilde{D} = \{d_1 \circ d_2 \circ \cdots \circ d_n(A): d = (d_1, d_2, \ldots, d_n)\}$ is a block (with respect to $R_{\theta_0}$ and $J_\lambda$) is a countable collection of sets contained in $A$, and that

$$A \supseteq d_1(A) \supseteq d_1 \circ d_2(A) \supseteq \cdots \supseteq d_1 \circ d_2 \circ \cdots \circ d_n(A).$$

Write $R$ for $R_{\theta_0}$ and $J$ for $J_\lambda$. The collection $\widetilde{D} = \{g_1, g_2\}$, where $g_1 = [J(A)] \cap W$, and $g_2 = [J(A), A \setminus J(A)] \cap W$, is a mutually disjoint cover of $W$ consisting of sets that are both open and closed in $W$. Now the homeomorphisms $J$ and $R$ induce homeomorphisms on $W$, which we also denote by $J$ and $R$ without loss of generality. Let

$$D_{\theta_0, \lambda, 2} = \{J(g_i) \cap W: i = 1, 2\} \cup \{R(g_i) \cap W: i = 1, 2\} = \{g_{i_1, i_2}: \langle i_1, i_2 \rangle \text{ is a 2-element sequence of 1's and 2's}\}.$$

Inductively, having defined the cover

$$D_{\theta_0, \lambda, l} = \{g_{i_1, i_2, \ldots, i_l}: \langle i_1, i_2, \ldots, i_l \rangle \text{ is an l-element sequence of 1's and 2's}\},$$

define

$$D_{\theta_0, \lambda, l+1} = \{J(g_{i_1, i_2, \ldots, i_l}) : g_{i_1, i_2, \ldots, i_l} \in D_{\theta_0, \lambda, l}\} \cup \{R(g_{i_1, i_2, \ldots, i_l}) : g_{i_1, i_2, \ldots, i_l} \in D_{\theta_0, \lambda, l}\}.$$

Without loss of generality, assume that for each $l$, $g_{i_1, i_2, \ldots, i_l} \subseteq g_{i_1, i_2, \ldots, i_l \ldots}$. Then for each finite sequence $\langle i_1, i_2, \ldots, i_l \rangle$, let

$$\tilde{g}_{i_1, i_2, \ldots, i_l} = \bigcap_{1 \leq i \leq l} g_{i_1, i_2, \ldots, i_l}.\$$

Hence, the collection $D_{\theta_0, \lambda} = \{\tilde{g}_{i_1, i_2, \ldots, i_l}: \langle i_1, i_2, \ldots, i_l \rangle \text{ is a finite sequence of 1's and 2's}\}$ is a basis of open and closed (and therefore compact) sets for the subspace $W$ of $F(A)$. We may, in addition, assume that for each $\tilde{g}_{i_1, i_2, \ldots, i_l} \in D_{\theta_0, \lambda}$, since $\tilde{g}_{i_1, i_2, \ldots, i_l} \supseteq d_i \circ d_{i_2} \circ \cdots \circ d_{i_l}(A)$ for some unique sequence $\langle d_i, d_{i_2}, \ldots, d_{i_l} \rangle$ of $R$'s and $J$'s, that whenever $i_j = 1$, $d_{i_j} = J$, and whenever $i_j = 2$, $d_{i_j} = R$. Then

$$\widetilde{D} = \{d_i \circ d_{i_2} \circ \cdots \circ d_{i_l}(A) \cap W: \langle d_i, d_{i_2}, \ldots, d_{i_l} \rangle \text{ is a finite sequence of R's and J's}\}$$

is a countable basis of open and closed (and therefore compact) sets for the subspace $W$. The remainder of this argument is a straightforward application of the topology on $W$ induced by the Hausdorff metric on this subspace of continua in $Y$, and we omit the proof. \(\square\)
Proposition 23. The countable collection $W_{\theta_0,\lambda}^Q$ of continua is dense in the collection $W_{\theta_0,\lambda}$ (in the Hausdorff metric sense).

Proof. Give $D_{\theta_0,\lambda}$ the meaning that it had in the previous argument, so that $D_{\theta_0,\lambda}$ is a countable basis of open and closed sets for the subspace $W_{\theta_0,\lambda}$ of $\mathcal{F}(A)$. Suppose that $e = \langle e_1, e_2, \ldots \rangle$ is in $\Sigma_{\theta_0,\lambda}$, and for each $n$, let $\sigma_n$ denote the periodic sequence $\langle e_1, e_2, \ldots, e_{n-1}, e_1, e_2, \ldots, e_{n-1}, \ldots \rangle$. Then the sequence $A, e_1(A), e_1 \circ e_2(A), \ldots$ converges to $W_e$ the sequence $diam(A), diam(e_1(A)), diam(e_1 \circ e_2(A)), \ldots$ converges to 0 (where the diameter is taken with respect to the Hausdorff metric); and for each $n$, $e_1 \circ e_2 \circ \cdots \circ e_{n-1}(A) \supseteq W_e \cup W_{\sigma_n}$. It follows that for $\varepsilon > 0$, there is $N$ such that if $n > N$, then the Hausdorff distance from $W_e$ to $W_{\sigma_n}$ is less than $\varepsilon$. The result follows. \(\square\)

Proposition 24. Suppose that $e = \langle e_1, e_2, \ldots \rangle$ and $e' = \langle e'_1, e'_2, \ldots \rangle$ are sequences in $\Sigma_{\theta_0,\lambda}$ such that for some $n$ and $n'$, $e_{n-i} = e'_{n'-i}$ for each $i$. Then $W_e \neq \emptyset$ if and only if $W_{e'} \neq \emptyset$.

Proof. Let $e'' = \langle e_n, e_{n-1}, \ldots \rangle = \langle e'_{n'}, e'_{n'-1}, \ldots \rangle$. Then $W_{e''} = e'_{n} \circ e''_{n-1} \circ \cdots \circ e'_{0}$ and $W_e = e_1 \circ e_2 \circ \cdots \circ e_{n+1}$ (for $W_e''$). Then $W_{e''} \neq \emptyset$ if and only if $W_{e'} \neq \emptyset$. \(\square\)

Remark 25. One important difference between the periodic case (the boundary continua $W_{\theta_0,\lambda}$—see Lemmas 5 and 6) and the random case (boundary continua in $W_{\theta_0,\lambda} \setminus W_{\theta_0,\lambda}^Q$) is that it can never happen that a member of $W_{\theta_0,\lambda}$ has interior outside of $A_0$, while it can happen that a member of $W_{\theta_0,\lambda} \setminus W_{\theta_0,\lambda}^Q$ has interior outside $A_0$. For example, if we take the sequence in $\Sigma_{\theta_0,\lambda}$ with $-1$ member $x_0$ and all other members $x_k$ (that is, we are considering the boundary continuum $W = J_1 \circ R_{\theta_0} \circ R_{\theta_0} \circ R_{\theta_0} \cdots (A)$), then $W \setminus A_0$ has interior which is exactly that portion of the ellipse $E_1 = J_1(D_1(0))$ which is outside $D_1(0)$. In fact, if $e = \langle e_1, e_2, \ldots \rangle$ is a sequence such that $W_e \neq \emptyset$, then if $d = \langle d_1, d_2, \ldots, d_m \rangle$ is a block, then the sequence $e' = \langle d_1, d_2, \ldots, d_m, e_{-1}, e_{-2}, \ldots \rangle$ also gives a boundary continuum $W_{e'}$ with interior, as does any sequence formed by lopping off an initial segment of $e$, i.e., if $k$ is a positive integer, then $W_{e_k}$, where $e = \langle e_{-k}, e_{-k-1}, \ldots \rangle$ has nonempty interior.

Theorem 26. Suppose $\lambda > 1$ and $\theta_0 > 0$. Let

$$G_{\theta_0,\lambda} = \{ W_e \in W_{\theta_0,\lambda} : W_e \text{ is an arclike, indecomposable continuum} \}.$$ 

Then $G_{\theta_0,\lambda}$ is not empty, and $G_{\theta_0,\lambda}$ is a dense, $G_\delta$-subset of $W_{\theta_0,\lambda}$. (Note that each member of $G_{\theta_0,\lambda}$ must therefore have empty interior.)
Proof. For each positive integer \( n \), let \( H_n = \{ d_1 \circ d_2 \circ \cdots \circ d_n : d = (d_1, d_2, \ldots, d_n) \) is a block with \( n \) elements \}. Then \( H_n \) is a collection of exactly \( 2^n \) homeomorphisms, and if \( \varepsilon > 0 \), there is some \( \varepsilon_n > 0 \) such that if \( C = \{ c_0, c_1, \ldots, c_m \} \) is a tiling chain in \( A \) of mesh less than \( \varepsilon_n \), then for each \( d_1 \circ d_2 \circ \cdots \circ d_n \) in \( H_n \), the collection
\[
d_1 \circ d_2 \circ \cdots \circ d_n(C) = \{ d_1 \circ d_2 \circ \cdots \circ d_n(c_i) : 0 \leq i \leq m \}
\]
is a tiling chain of mesh less than \( \varepsilon \). Let \( R = R_{\theta_0} \) and \( J = J_\lambda \).

Choose a positive integer \( k \) so that \( k \theta_0 \) is not an odd multiple of \( \pi/2 \), but \( k \theta_0 > \pi/2 \). Let \( R^k = \tilde{R} \). There is some \( l \) such that \( J^l \circ \tilde{R} \circ J^l(A) \cap A_0 \) has the same number of components as the intersection of \( \tilde{R}(\mathbb{R}_x) \) with the \( y \)-axis, and each component of \( J^l \circ \tilde{R} \circ J^l(A) \cap A_0 \) extends from one vertical boundary of \( A_0 \) to the other and does not intersect either horizontal boundary. Let \( J^l = \tilde{J} \), and let \( \tilde{J} \circ \tilde{R} \circ \tilde{J} = g \). (See the proof of Theorem 13 if more details are needed. The idea is that \( \tilde{R}(\mathbb{R}_x) \cap \mathbb{R}_y \) has a positive odd number of components. Applying \( J^l \) to the set \( A \) first flattens and thins it, so that it approximates the set \( \mathbb{R}_x \), then \( \tilde{R} \) "stirs" this set \( J^l(A) \) in the unit disk, and then \( J^l \) flattens and squeezes much of the set \( \tilde{R}, J^l(A) \) out of the unit disk, so that the resulting set \( \tilde{J} \circ \tilde{R} \circ \tilde{J}(A) \) has the same number of components as \( \tilde{R}(\mathbb{R}_x) \cap \mathbb{R}_y \), but in addition, stretches from one vertical boundary of \( A_0 \) to the other and does not intersect either horizontal boundary.)

For each \( \tilde{d} = d_1 \circ d_2 \circ \cdots \circ d_n \) in \( H_n \), consider the composition \( \tilde{d} \circ g \). Note that \( \tilde{d} \circ g(A) \cap A_0 \) has at least three components, and has at least the same number of components as \( g(A) \cap A_0 \), since

(1) \( R \) does not change \( \partial A_0 \), while it does stir points inside \( A_0 \), and

(2) applying \( J \) can squeeze some the complication added by applying the \( R \)'s outside \( A_0 \), but it adds no complication itself.

Let \( P_R = \{(x_1, x_2) \in Y : x_1 \geq -1\} \), \( P_L = \{(x_1, x_2) \in Y : x_1 \leq 1\} \), and \( P_C = P_R \cap P_L \). There are

(1) a component \( \mathcal{C}_{\infty+} \) of \( P_L \cap (\tilde{d} \circ g(A)) \) that contains the point \( \infty_+ \),

(2) a component \( \mathcal{C}_{\infty-} \) of \( P_R \cap (\tilde{d} \circ g(A)) \) that contains the point \( \infty_- \), and

(3) a component \( \mathcal{C}_0 \) of \( P_C \cap (\tilde{d} \circ g(A)) \) that contains the point \( 0 \).

Further, no two of the components \( \mathcal{C}_{\infty+}, \mathcal{C}_{\infty-}, \) and \( \mathcal{C}_0 \) intersect.

There is some integer \( k_{n,1} \) such that \( \tilde{J}^{k_{n,1}}(A) \) can be partitioned into a closed chain
\[
C_{n,1} = \{ c(n, 1, 0), c(n, 1, 1), \ldots, c(n, 1, \alpha_{n,1}) \}
\]
with links \( c(n, 1, i) \) of such small diameter that the mesh of the closed chain \( g(C_{n,1}) = \{ g(c(n, 1, i)) : 0 \leq i \leq \alpha_{n,1} \} \) is less than \( \varepsilon_n \). Thus, the mesh of each tiling chain \( \tilde{d} \circ g(C_{n,1}) = \{ \tilde{d} \circ g(c(n, 1, i)) : 0 \leq i \leq \alpha_{n,1} \} \) is less than \( \varepsilon \).

Recall that the collection \( D_{\theta_0, \lambda} \) defined in Proposition 22 gives a countable basis of open and closed sets for \( W_{\theta_0, \lambda} \). Each \( \tilde{g}_{i_1, i_2, \ldots, i_l} \) in \( D_{\theta_0, \lambda} \) is associated with the unique composition \( d_{i_1} \circ d_{i_2} \circ \cdots \circ d_{i_l} \), where \( d_{i_l} = J \) if \( i_l = 1 \) and \( d_{i_l} = R \) if \( i_l = 2 \), in the sense that \( \tilde{g}_{i_1, i_2, \ldots, i_l} \supseteq d_{i_1} \circ d_{i_2} \circ \cdots \circ d_{i_l}(A) \). It follows that
\[
\widetilde{D}_{\theta_0, \lambda} = \{ d_1 \circ d_2 \circ \cdots \circ d_n(A) \cap W_{\theta_0, \lambda} : n \text{ is a positive integer}, d_i = R \text{ or } d_i = J \text{ for each } i \text{ such that } 1 \leq i \leq n \}
\]
is a countable basis of open and closed sets for $W_{\theta_0, \lambda}$. Then for each $d_1 \circ d_2 \circ \cdots \circ d_n(A) - \hat{d}(A)$ in $\widehat{D}_{\theta_0, \lambda}$, there is some finite composition $g_{\hat{d}}$ of $J$’s and $R$’s such that

1. $\hat{d} \circ g_{\hat{d}}(A)$ is a closed (nonempty) subset of $\hat{d}(A)$ and $[\hat{d} \circ g_{\hat{d}}(A)] \cap W_{\theta_0, \lambda}$ is open in the space $W_{\theta_0, \lambda}$;
2. $\hat{d} \circ g_{\hat{d}}(A) \cap A_0$ has at least three components;
3. there is a component $\mathcal{C}_{\infty_+^x}$ of $P_L \cap (\hat{d} \circ g_{\hat{d}}(A))$ that contains the point $\infty_+^x$;
4. there is a component $\mathcal{C}_{\infty_-^x}$ of $P_R \cap (\hat{d} \circ g_{\hat{d}}(A))$ that contains the point $\infty_-^x$;
5. there is a component $\mathcal{C}_0$ of $P_C \cap (\hat{d} \circ g_{\hat{d}}(A))$ that contains the point 0;
6. no two of the components $\mathcal{C}_{\infty_+^x}$, $\mathcal{C}_{\infty_-^x}$, and $\mathcal{C}_0$ intersect; and
7. $\hat{d} \circ g_{\hat{d}}(A)$ admits a tiling chain of mesh less than $\frac{1}{2}$.

Then if $G_1 = \{ [\hat{d} \circ g_{\hat{d}}(A)] \cap W_{\theta_0, \lambda} : \hat{d}(A) \in \widehat{D}_{\theta_0, \lambda} \}$, $\bigcup G_1$ is an open dense subset of $W_{\theta_0, \lambda}$. (Note that for $\hat{d} = d_1 \circ d_2 \circ \cdots \circ d_n \in H_n$, $g_{\hat{d}} = g \circ J_{n-1}$.)

Now $\tilde{R} \circ \tilde{J} = g$ was chosen so that $g(A) \cap A_0$ has the same number of components as the intersection of $R(\mathbb{R}_+) \cap \mathbb{R}_x$ by the $y$-axis, and each component of $(g(A) \cap A_0$ extends from one vertical boundary of $A_0$ to the other and does not intersect either horizontal boundary. Further, $\kappa_{n,1}$ was chosen so that $J^{\kappa_{n,1}}(A)$ can be written as a closed chain $C_{n,1} = \{c(n, 1, 0), c(n, 1, 1), \ldots, c(n, 1, \alpha_{n,1})\}$ with links $c(n, 1, i)$ of such small diameter that the mesh of the closed chain $g(C_{n,1}) = \{g(c(n, 1, i)) : 0 \leq i \leq \alpha_{n,1}\}$ is sufficiently small that the mesh of each component of $g(C_{n,1})$ has at least nine components:
- there is a component $\mathcal{C}_{\infty_+^x}$ of $P_L \cap (\hat{d} \circ g_{\hat{d}} \circ g(A))$ that contains the point $\infty_+^x$;
- there is a component $\mathcal{C}_{\infty_-^x}$ of $P_R \cap (\hat{d} \circ g_{\hat{d}} \circ g(A))$ that contains the point $\infty_-^x$;
- there is a component $\mathcal{C}_0$ of $P_C \cap (\hat{d} \circ g_{\hat{d}} \circ g(A))$ that contains the point 0; and
- no two of the components $\mathcal{C}_{\infty_+^x}$, $\mathcal{C}_{\infty_-^x}$, and $\mathcal{C}_0$ intersect.

For each $n$, there is some integer $\kappa_{n,2}$ such that $J^{\kappa_{n,2}}(A)$ can be written as a closed chain $C_{n,2} = \{c(n, 2, 0), c(n, 2, 1), \ldots, c(n, 2, \alpha_2)\}$ with links $c(n, 2, i)$ of such small diameter that the mesh of the closed chain

$\hat{d} \circ g_{\hat{d}} \circ g(C_{n,2}) = \{\hat{d} \circ g_{\hat{d}} \circ g(c(n, 2, i)) : 0 \leq i \leq \alpha_{n,2}\}$

is sufficiently small that the mesh of each component of $g(C_{n,2})$ has at least nine components:

$\hat{d} \circ g(C_{n,1}) = \{\hat{d} \circ g(c(n, 1, i)) : 0 \leq i \leq \alpha_{n,1}\}$

is less than $\frac{1}{4}$. Let $g_{\hat{d}} \circ g \circ J^{\kappa_{n,2}} = g_{\hat{d},2}$ (so that for each $\hat{d} \in H_n$, $g_{\hat{d},2} = g \circ J^{\kappa_{n,1}} \circ g \circ J^{\kappa_{n,2}}$).

Thus, for each $\hat{d}(A)$ in $\widehat{D}_{\theta_0, \lambda}$, there is some finite composition $g_{\hat{d},2}$ of $J$’s and $R$’s such that

1. $\hat{d} \circ g_{\hat{d},2}(A)$ is an open and closed (nonempty) subset of $[\hat{d}(A)] \cap W_{\theta_0, \lambda}$;
2. $\hat{d} \circ g_{\hat{d},2}(A) \cap A_0$ has at least nine components;
3. $\hat{d} \circ g_{\hat{d},2}(A) \cap (A_0 \cup A_1)$ has at least three components;
4. there is a component $\mathcal{C}_{\infty_+^x}$ of $P_L \cap (\hat{d} \circ g_{\hat{d},2}(A))$ that contains the point $\infty_+^x$. 

(5) there is a component $C_{\infty_x^+}$ of $P_{R} \cap (\widehat{d} \circ g_{\delta,2}(A))$ that contains the point $\infty_x^+$;
(6) there is a component $C_{0}$ of $P_{C} \cap (\widehat{d} \circ g_{\delta,2}(A))$ that contains the point 0;
(7) no two of the components $C_{\infty_x^+}, C_{\infty_x^-},$ and $C_0$ intersect; and
(8) $\widehat{d} \circ g_{\delta,2}(A)$ admits a tiling chain of mesh less than $1/4$.

Then, if $G_2 = \{ [\widehat{d} \circ g_{\delta,2}(A)] \cap W_{\theta_0,\lambda} : \widehat{d}(A) \text{ is in } D_{\theta_0,\lambda} \}, \bigcup G_2$ is an open dense subset of $W_{\theta_0,\lambda}$.

We can continue this process. At the $m$th step we obtain a finite composition $g_{d,m}$ of $J$’s and $R$’s such that

1. $\widehat{d} \circ g_{d,m}(A)$ is an open and closed (nonempty) subset of $[\widehat{d}(A)] \cap W_{\theta_0,\lambda};$
2. $\widehat{d} \circ g_{d,m}(A) \cap A_0$ has at least $3^m$ components (each stretching from vertical boundary to vertical boundary of $A_0$),
3. $\widehat{d} \circ g_{d,m}(A) \cap (A_0 \cup A_1 \cup \cdots \cup A_{m-1})$ has at least three components (each stretching from far vertical boundary to far vertical boundary of $A_0 \cup A_1 \cup \cdots \cup A_{m-1}$);
4. there is a component $C_{\infty_x^+}$ of $P_{L} \cap (\widehat{d} \circ g_{d,m}(A))$ that contains the point $\infty_x^+$;
5. there is a component $C_{\infty_x^-}$ of $P_{L} \cap (\widehat{d} \circ g_{d,m}(A))$ that contains the point $\infty_x^-$;
6. $\widehat{d} \circ g_{d,m}(A)$ has a tiling chain of mesh less than $1/2^m$.

Then, if $G_m = \{ [\widehat{d} \circ g_{d,m}(A)] \cap W_{\theta_0,\lambda} : \widehat{d}(A) \text{ is in } D_{\theta_0,\lambda} \}, \bigcup G_m$ is an open dense subset of $W_{\theta_0,\lambda}$.

Let $G_{\theta_0,\lambda} = G = \bigcap_{m \geq 1}(\bigcup G_n)$, so that $G_{\theta_0,\lambda}$ is a dense $G_\delta$-subset of $W_{\theta_0,\lambda}$. It follows from the construction that each member $W_e$ of $G_{\theta_0,\lambda}$ is arclike (because of the application of carefully chosen, but arbitrarily large compositions of $J$’s, which mean that each point (continuum) in $G_{\theta_0,\lambda}$ has a chain cover of mesh less than $1/2^m$). It also follows that the composant of $W_e$ that contains $\infty_x^+$ is disjoint from the composant of $W_e$ that contains $\infty_x^-$, because of the folding introduced first by the application of a composition of $R$’s, and then squeezed out of the unit disk and “thinned” to the desired width by an application of $J$’s. Thus, $W_e$ is an indecomposable, arclike continuum, and $(W_e)^\circ$ is empty. □

**Theorem 27.** Suppose $\lambda > 1$ and $\theta_0 > 0$. Let

$$M_{\theta_0,\lambda} = \{ W_e \text{ in } W_{\theta_0,\lambda} : e = \langle e_1, e_2, \ldots \rangle \text{ contains an infinite number of copies of all blocks of } J_\lambda \text{'s and } R_{\theta_0} \text{'s} \},$$

and let $\widehat{M}_{\theta_0,\lambda} = \{ e \in \Sigma_{\theta_0,\lambda} : W_e \in M_{\theta_0,\lambda} \}$. Then $\widehat{M}_{\theta_0,\lambda}$ has measure 1 in $\Sigma_{\theta_0,\lambda}$, and each member $W_e$ of $M_{\theta_0,\lambda}$ has empty interior and contains an indecomposable continuum which contains the points 0, $\infty_x^-$ and $\infty_x^+$.  

**Proof.** Let $J = J_\lambda$ and $R = R_{\theta_0}$. As in the last proof, we use the collection
\[
\widetilde{D}_{\theta_0, \lambda} = \{ [d_1 \circ d_2 \circ \cdots \circ d_n(A)] \cap W_{\theta_0, \lambda} : d = \langle d_1, d_2, \ldots, d_n \rangle \}
\]

Since this collection is a countable basis of open and closed sets for \( W_{\theta_0, \lambda} \). As before, denote each \( d_1 \circ d_2 \circ \cdots \circ d_n(A) \) as \( \hat{d}(A) \). Note first that since for each \( m \), there exist blocks of the form \( d = \langle d_1, d_2, \ldots, d_m \rangle \) with \( d_i = J \) for each \( i \), no \( W_e \) in \( M_{\theta_0, \lambda} \) can have interior: if \( W_e \) in \( M_{\theta_0, \lambda} \) has interior, then for some \( i \),

\[
o = \left( \bigcup_{l=0}^{i} A_l \right) \cap W_e \neq \emptyset
\]

and \( o \) has some positive measure \( \alpha \). For each \( j \), let \( o_j = (e^{-1} \circ e^{-2} \circ \cdots \circ e^{-j})^{-1}(o) \). Thus, \( \mu_A(o_j) = \alpha \) for each \( j \). Now

\[
\mu_A \left( \bigcup_{l=0}^{i} A_l \right) = \beta > 0,
\]

and it is possible to choose \( m \) so large that \( \mu_A(J^m(\bigcup_{l=0}^{j} A_l) \cap (\bigcup_{l=0}^{i} A_l)) < 3\alpha/4 \). There is some \( \kappa \) such that \( e_{-k-j} = J \) for \( 1 \leq j \leq m \). Now \( W_{e'} \), where \( e' = (e_{-k-m-1} \circ e_{-k-m-2} \circ \cdots) \), is a continuum in \( A \), and

\[
\mu_A(o_k) \leq \mu_A \left( J^m \left( \bigcup_{l=0}^{i} A_l \right) \cap \left( \bigcup_{l=0}^{i} A_l \right) \right)
\]

since

\[
o_k \subseteq J^m \left( \bigcup_{l=0}^{i} A_l \right) \cap \left( \bigcup_{l=0}^{i} A_l \right).
\]

But then \( \alpha = \mu_A(o_k) < \mu_A(J^m(\bigcup_{l=0}^{j} A_l) \cap (\bigcup_{l=0}^{i} A_l)) < 3\alpha/4 \), and this is a contradiction.

Choose a positive integer \( k \) so that \( k\theta_0 \) is not an odd multiple of \( \pi/2 \), but \( k\theta_0 > \pi/2 \). Let \( R^k = \hat{R} \). There is some \( l \) such that \( J^l \circ \hat{R} \circ J^l(A) \cap A_0 \) has the same number of components as the intersection of \( \hat{R}(\mathbb{R}_x) \) with the \( y \)-axis, and each component of \( J^l \circ \hat{R} \circ J^l(A) \cap A_0 \) extends from one vertical boundary of \( A_0 \) to the other and does not intersect either horizontal boundary. Let \( J^l = \hat{J} \), and let \( J \circ \hat{R} \circ \hat{J} = g \).

Note that blocks associated with the composition \( g \) occur in each sequence \( e \) such that \( W_e \) is in \( M_{\theta_0, \lambda} \). Let \( P_R = \{(x_1, x_2) \in Y : x_1 \geq -1 \} \), \( P_L = \{(x_1, x_2) \in Y : x_1 \leq 1 \} \), and \( P_C = P_R \cap P_L \). Roughly, if \( h \) is any finite composition of \( J \)'s and \( R \)'s, then \( h \circ g(A) \cap A_0 \) can be rather more complicated topologically than is \( g(A) \cap A_0 \), but it cannot simplify the set \( g(A) \cap A_0 \) -- it just adds more squeezing and winding. To be more precise: \( h \circ g(A) \cap A_0 \) has at least three components, and

1. there is a component \( C_{x_+^+} \) of \( P_L \cap (d \circ g_{d,2}(A)) \) that contains the point \( \infty_x^+ \);
2. there is a component \( C_{x_-^+} \) of \( P_R \cap (\hat{d} \circ g_{d,2}(A)) \) that contains the point \( \infty_x^- \);
3. there is a component \( C_0 \) of \( P_C \cap (d \circ g_{d,2}(A)) \) that contains the point \( 0 \); and
4. no two of the components \( C_{x_+^+}, C_{x_-^+}, \) and \( C_0 \) intersect.
Likewise, if \( h \) is any finite composition of \( J \)'s and \( R \)'s, then
1. \( h \circ g^2(A) \cap A_0 \) has at least nine components;
2. \( h \circ g^2(A) \cap (A_0 \cup A_1) \) has at least three components;
3. there is a component \( C_{\infty_x^+} \) of \( P_L \cap (\hat{d} \circ g_{d,2}(A)) \) that contains the point \( \infty_x^+ \);
4. there is a component \( C_{\infty_x^-} \) of \( P_R \cap (\hat{d} \circ g_{d,2}(A)) \) that contains the point \( \infty_x^- \);
5. there is a component \( C_0 \) of \( P_C \cap (\hat{d} \circ g_{d,2}(A)) \) that contains the point 0; and
6. no two of the components \( C_{\infty_x^+} \), \( C_{\infty_x^-} \), and \( C_0 \) intersect.

We can continue this process. At the \( m \)th step we obtain for each finite composition \( h \) of \( J \)'s and \( R \)'s, the finite composition \( h \circ g^m \) with the following properties:
1. \( h \circ g^m(A) \cap A_0 \) has at least \( 3^m \) components (each stretching from vertical boundary to vertical boundary of \( A_0 \));
2. \( h \circ g^m(A) \cap (A_0 \cup A_1 \cup \cdots \cup A_{m-1}) \) has at least three components (each stretching from far vertical boundary to far vertical boundary of \( A_0 \cup A_1 \cup \cdots \cup A_{m-1} \));
3. there is a component \( C_{\infty_x^+} \) of \( P_L \cap (\hat{d} \circ g_{d,2}(A)) \) that contains the point \( \infty_x^+ \);
4. there is a component \( C_{\infty_x^-} \) of \( P_R \cap (\hat{d} \circ g_{d,2}(A)) \) that contains the point \( \infty_x^- \);
5. there is a component \( C_0 \) of \( P_C \cap (\hat{d} \circ g_{d,2}(A)) \) that contains the point 0; and
6. no two of the components \( C_{\infty_x^+} \), \( C_{\infty_x^-} \), and \( C_0 \) intersect.

Each \( \mathcal{W}_e \) in \( M_{\theta_0,\lambda} \) is derived from the sequence \( e \) which contains blocks of the form \( h \circ g^m \), and, in fact, \( e \) begins with an initial segment corresponding to the composition \( h \circ g^m \). Also, \( \mathcal{W}_e \) contains a continuum \( \mathcal{W}_e'' \) irreducible from \( \infty_x^+ \) to 0. However, because of these initial segments, such a continuum must also contain \( \infty_x^- \). Then, again because of these initial segments, the continuum \( \mathcal{W}_e'' \) must be irreducible between \( \infty_x^+ \) and \( \infty_x^- \) and 0. Then \( \mathcal{W}_e'' \) is indecomposable, and the proof is finished. \( \Box \)

**Remark.** It is probably the case that each member of \( M_{\theta_0,\lambda} \) is indecomposable. The sticking point seems to be obtaining the irreducibility of each \( \mathcal{W}_e \).

**References**