Basins of Wada*

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We describe situations in which there are several regions (more than two) with the Wada property, namely that each point that is on the boundary of one region is on the boundary of all. We argue that such situations arise even in studies of the forced damped pendulum, where it is possible to have three attractor regions coexisting, and the three basins of attraction have the Wada property.

1. Introduction

We are accustomed to the idea that while the boundary between two regions is likely to consist of arcs, the points that are in the boundary of three or more regions are intuitively expected to be isolated. In the United States, there is only one point that is in the boundary of four states. In dynamical systems, the boundaries are more complicated.

Topologists discovered that it is possible to have three or more regions $B_i$, each an open set such that if $p$ is in the boundary of any one $B_i$, then it is in fact in the boundary of all the sets $B_i$. Such a possibility is hard to imagine. We will say that the sets $B_i$ satisfy the \textit{Wada property} if each boundary point of any $B_i$ is in the boundary of all the $B_i$. The first explicit example \cite{22} was attributed to Yoneyama. See also ref. \cite{15}. We begin in section 2 by describing the example called "The Lakes of Wada". The main point of this paper is that the Wada property may be quite common in dynamics. In section 3 we give an example using the Poincaré return map of the forced damped pendulum, choosing parameters so that there are four attracting fixed points. We argue that their basins of attraction satisfy the Wada property.

One of the objectives of science is prediction. For a dynamical system with several attractors, the main prediction problem then is to determine which basin a given point is in, that is, which attractor the trajectory through that point will be attracted to.

When boundaries are complicated in a specified bounded region of initial points \cite{16}, a small uncertainty in the position of the initial point may yield a large uncertainty as to which attractor the trajectory will go to. Consider a possible, but incorrect analogy: in the United States, one might not know which state a point is in, if there is some uncertainty in the position, but the list of possible states would be small. We however report on situations where if there is any uncertainty as to which basin the point is in, then in fact the point might be in any of the basins. None can be ruled out because any point on the boundary is arbitrarily close to points in all the basins.

When a collection of open sets $B_i$ satisfies the Wada property, the boundary $W$ (for Wada) of those sets must have strange topological properties. Specifically, such a set must either be "indecomposable" or must be the union of two

\*This research was supported in part by NSF grant DMS-9006931 and AFOSR grant 81-0217.
indecomposable sets [15]. Section 4 describes indecomposable sets. (Readers not familiar with terms such as connected, arc, component, and continuum may want to begin with section 4.) These are connected sets which have the property that if one attempts to cut them in two, the cut necessarily splits the set into infinitely many connected sets. In fact many other sets that arise in dynamics in the plane are indecomposable, including many chaotic attractors.

While we have become accustomed to finding Cantor-like sets in dynamics, it appears that much stranger sets may be common. These “hereditarily indecomposable sets” are very briefly discussed in section 5.

2. The lakes of Wada example

The following example shows how it is possible to have three regions with the Wada property. Each point that is in the boundary of any of the regions is in fact in the boundary of all three. It is easy to extend the construction to more than three regions. The regions in question are topologically equivalent to disks. The construction looks extremely artificial, yet we will see that such regions occur naturally in dynamical systems of the plane.

Think of a spherical planet covered with one ocean filled with red water, except for a single island, a disk D. Further, there are two lakes E and F on the island D, the lake E being filled with blue water and the lake F with green water. The construction consists of digging a series of canals, eventually carving away almost all the land.

The first step is to begin digging canals on the island D as follows.

1. (See fig. 1.) Between time \( t = 0 \) and \( t = \frac{1}{2} \), dig a canal from the ocean into the land so that once the canal is dug, no point of land is at a distance of more than 1 unit from red water.

2. Between time \( t = \frac{1}{2} \) and \( t = \frac{3}{4} \), dig a presumably thinner canal from the blue lake E into the remaining land on D so that each point of the land remaining after the canal is dug is at a distance of no more than \( \frac{1}{2} \) unit from blue water.

3. Between times \( t = \frac{3}{4} \) and \( t = \frac{7}{8} \), dig a third canal from the green lake into the remaining land (which now consists of fairly thin strips) so that after the third canal is dug, each point of land is at a distance of no more than \( \frac{1}{4} \) unit from green water.

Between times \( t = \frac{7}{8} \) and \( \frac{15}{16} \), work resumes on the first canal, this time extending it so that each point of remaining land will be within a distance of \( \frac{1}{8} \) unit from red water. Continue digging between the three canals. At the limiting time \( t = 1 \), the land remaining, W, is just the boundary of canals and an indecomposable continuum. The land W is a continuum because it is the intersection of a nested collection of continua (the land remaining at each stage of the construction is a continuum). Further, with some care taken in the digging of the canals, we may guarantee that if S is a small closed disk, whose interior W intersects, then \( S \cap W \) consists of an uncountable collection of arcs (plus possibly some points on the boundary of S). In fact, W has the following
property which is known to be equivalent to being indecomposable: For every closed set \( T \) that does not contain all of \( W \), but does contain a closed disk \( S \) whose interior intersects \( W \), the intersection \( T \cap W \) will have uncountably many components. For the continuum \( W \), each component of \( T \cap W \) is either an arc or a single point.

This continuum sits in the plane in a very interesting way. Topologically, the red water region before the canal construction began (the ocean), and after it was completed (the ocean and a very long canal) are equivalent. The same is true for the blue water regions (a lake before and a lake and a canal after) and the green water regions. However, the topological nature of the land changed dramatically as a result of the canal construction. In fact, looking inside any small closed disk \( S \) whose interior intersects \( W \), one will see an infinite number of red water regions, an infinite number of blue water regions and an infinite number of green water regions, all separated by "infinitely thin" strips of land. It is not possible to choose the disk \( S \) so that its interior intersects \( W \) and contains only blue water, for example. Any such disk must contain blue water, green water and red water. If one considers the blue water region, say, after the construction, and takes the closure of this region, one gets in that closure all of \( W \). The same is true of the red water and green water regions.

Kan [14] used a similar construction in creating a dynamical system with a complicated attractor.

### 3. Basins

For maps in the plane, in cases where there are more than two attractors, we sometimes seem to see the boundary of the basins satisfying the Wada property: each boundary point has sections of each basin arbitrarily close to it. The following proposition shows why this can occur.

Let \( F: \mathbb{R}^2 \to \mathbb{R}^2 \) be a diffeomorphism with a saddle fixed point \( p (F(p) = p) \). Define the unstable and stable manifolds of \( p \):

\[
\Lambda^u(p) = \{ q \in \mathbb{R}^2 | \text{the sequence } \{ F^{-n}(q) \} \text{ converges to } p \},
\]

\[
\Lambda^s(p) = \{ q \in \mathbb{R}^2 | \text{the sequence } \{ F^{n}(q) \} \text{ converges to } p \}.
\]

For an open set \( B \), write \( \partial B = \overline{B} - B \) for the boundary of \( B \).

**Proposition 1.** Let \( B \subseteq \mathbb{R}^2 \) be an open invariant set and let \( p \) be a point not in \( B \) whose unstable manifold \( \Lambda^u(p) \) intersects \( B \). Then the closure of the stable manifold \( \Lambda^s(p) \) of \( p \) is a subset of \( \partial B \).

**Proof.** The proof follows from standard arguments. See the "Lambda lemma" (ref. [12], p. 247). Fig. 2 illustrates the ideas. Since the unstable manifold of \( p \) intersects \( B \), we can choose a disk \( D \) in \( B \). Then the images \( F^{-n}(D) \) of \( D \) get arbitrarily close to each point of \( \Lambda^u(p) \) as \( n \) increases, and therefore arbitrarily close to each point of \( \overline{\Lambda^u(p)} \). In other words, for each point of \( q \) in the stable manifold \( \Lambda^s(p) \), there exist points \( x_n \in D \) such that the sequence \( \{ F^{-n}(x_n) \} \) converges to \( q \). Since \( B \) is invariant, \( F^{-n}(x_n) \) is in \( B \), so \( q \in \overline{B} \). \( \square \)
Let the sets $B_1, \ldots, B_k$ contained in $\mathbb{R}^2$ be basins of attraction (or just any open invariant sets). Fig. 3 illustrates proposition 2.

**Proposition 2.** If $\Lambda^u(p)$ intersects all the sets $B_1, \ldots, B_k$, then each $q$ in $\Lambda^u(p)$ is in the boundary $\partial B_i$ of $B_i$ for each $i$.

The following theorems are applicable here, too. The first theorem is due to Barge [1] and appears here specialized for our situation. The second theorem is due to Kuratowski [15] and dates back to 1924.

**Barge’s theorem.** Let $F$ be a dissipative diffeomorphism on $\mathbb{R}^2$ (i.e. $|\det F(x)| < 1$ for all $x$). Assume $p$ is a saddle fixed point whose stable and unstable manifolds intersect at some point $q \neq p$. Assume the unstable manifold is bounded (and has closure $Q$). Then $Q$ is an indecomposable continuum.

**Kuratowski’s theorem.** Let $B_1, \ldots, B_k$ be regions in $\mathbb{R}^2$ with $k \geq 3$. If $\partial B_1 = \partial B_2 = \ldots = \partial B_k$, then $\partial B_i$ is either an indecomposable continuum or the union of two indecomposable continua.

$Q$ is a basic set if $Q$ is compact and (1) there is a dense trajectory in $Q$ (there is some $q \in Q$ for which $\{F^n(q)\}_{n \geq 0}$ has closure equal to $Q$), and (2) $Q$ is maximal with respect to (1); that is, there is no strictly larger compact set $Q$, containing $Q$ that has a dense trajectory. The basic set is plotted in green in plate 1.

The equation

$$\frac{d^2 \theta}{dt^2} + \frac{1}{10} \frac{d\theta}{dt} + \sin \theta = \frac{7}{4} \cos t$$

has (at least) four attracting periodic orbits (each having period $2\pi$). All the numerical computations were made using a fourth order Runge–Kutta method with fixed time step of size $2\pi / 100$. For the time $2\pi$ Poincaré return map, these orbits are attracting fixed points. The four crosses in plate 2 show their locations. Experimentation reveals that points in the $(\theta, d\theta/dt)$ space that are within 0.01 of the four attracting points are attracted to those points. Each of the pictures is plotted with $d\theta/dt$ vertically on a scale $-2 < d\theta/dt < 4$ and $\theta$ horizontally, $-\pi < \theta < \pi$, using an $850 \times 600$ point grid. Plates 1 and 2 are created by first showing which initial points $(\theta, d\theta/dt)$ with initial time $t = 0$ eventually come within 0.01 of each of the attractors. The points going to each of the attractors are color coded red, blue, yellow, and green, respectively. There are additional attractors, which together attract less than 0.05% of the points in the picture. These are likely to have the same properties as the other four, but it is difficult to tell.

The “red” attractor corresponds to a periodic orbit of the pendulum for which $\theta(t)$ increases, $\theta(t + 2\pi) = \theta(t) + 2\pi$ for all $t$, so that the pendulum swings clockwise through $360^\circ$ each period. The blue attractor swings counterclockwise with $\theta(t + 2\pi) = \theta(t) - 2\pi$. The green and yellow attractors are bounded with $\theta(t + 2\pi) = \theta(t)$. The union of the green and the yellow attractor regions forms one Wada region. For all of the attractors $d\theta/dt(t)$ is periodic with period $2\pi$. All the attractors are periodic when $\theta(t)$ is calcu-
lated mod 2π, but if θ(t) is instead viewed on the real line, two of the periodic orbits are no longer periodic.

The two pictures additionally use a saddle periodic point having period 2 for the time 2π Poincaré return map. The position of the period 2 orbit is shown in plate 1 by two white crosses. Plate 2 shows a short segment of the unstable manifold of the period 2 point, clearly crossing all four basins. (This saddle was chosen somewhat at random, and other periodic saddles would presumably have similar behavior.) Plate 1 shows an extensive part of the stable manifold of the orbit. The chunk of stable manifold plotted suggests that the entire stable manifold is dense in the boundary of the three regions. Since the unstable manifold enters each of the three basins, the proposition can be applied to each of the basins. Of course, we consider the yellow and green regions to be a single (generalized) basin.

We believe the stable manifold is dense in the boundary in this example. The method of computing stable and unstable manifolds is quite accurate as reported in ref. [23], and is in effect limited only by the accuracy of numerical solver (the Runge-Kutta solver) used for evaluating the map.

4. Indecomposable continua

Most scientists working in dynamics are aware that the limit sets associated with dynamical systems, even “nice” ones, can be quite complicated. This section reports on the topological structures of dynamics and shows how far from being “somewhat Euclidean” these sets can be. It also gives an introduction to indecomposable continua, in order to give examples of how they can arise in dynamics, and in order to discuss our recent applications of these ideas to dynamics.

The bad news is that indecomposable continua require a different intuition than that of most mathematicians and scientists. The good news is that indecomposable continua have been studied since the early part of this century, and there exists a considerable body of literature of them. Also, even though they must be dealt with on their own terms, they do have structure. Often that structure is quite rich and very strong rules govern their behavior.

A closed set A in the plane, or in any topological space, is said to be connected if it cannot be written as the union of two disjoint, nonempty closed sets. A continuum K in ℝ^n is a compact, connected subset of ℝ^n. In this paper, the reader can consider all sets and spaces to be subsets of ℝ^n for some n. The cylinder, for example, can be viewed as a subset of ℝ^3. If X and Y are spaces or subsets of a space such as ℝ^n, h: X → Y is one-to-one, continuous and onto, and h^{-1} is continuous, then h is a homeomorphism. The sets X and Y in this case are said to be homeomorphic, or topologically equivalent. An arc is a set that is homeomorphic to the unit interval. A set X is said to be arcwise connected if for each pair p, q of points in X, there is an arc contained in X that contains both p and q. If Y and X are continua, and Y ⊆ X, then Y is a subcontinuum of X. If Y ≠ X, then Y is a proper subcontinuum of X.

(1) A remark and an example. Every continuum in ℝ^1 is an arc or a point (and arcs in ℝ^n are compact intervals), and therefore every continuum in ℝ^1 is arcwise connected. However, this is not true for subsets of higher-dimensional spaces. The simplest example of a continuum that is not arcwise connected is the “topologist’s sin(1/x) curve” X in the plane. Define X_0 to be the graph \{(x, y) ∈ ℝ^2 | 0 < x ≤ 1 \text{ and } y = \sin(1/x)\} and X_1 to be the vertical line \{(x, y) | x = 0 \text{ and } y ∈ [-1, 1]\}. Then X_1 is in the closure of X_0, and X is the continuum X_1 ∪ X_0 pictured in fig. 4. Note that (0, 1) and (1, sin 1) are in X, but there is no arc from (0, 1) to (1, sin 1) that is contained in X.

(2) Remark. Any open, connected subset of ℝ^n is arcwise connected.
A set \( X \) is said to be \textit{locally connected at the point} \( p \) in \( X \), if for each open set \( U \) that contains \( p \), there is an open set \( V \) containing \( p \) such that \( V \) is contained in \( U \), and \( V \cap X \) is connected. \( X \) is \textit{locally connected} if it is locally connected at each of its points.

(3) \textit{Example}. Begin with a Cantor set lying in the segment from \((0,0)\) to \((1,0)\) in the plane. The Cantor fan consists of that Cantor set plus the point at \((\frac{1}{2},1)\) plus the line segments that run from each point of the Cantor set to \((\frac{1}{2},1)\).

The Cantor fan (fig. 5) is a continuum in \( \mathbb{R}^2 \) which is arcwise connected, but is not locally connected. (It is locally connected at the point \((\frac{1}{2},1)\), but is not locally connected at any other point.) The topologist’s \( \sin(1/x) \) curve is locally connected at each point of \( X_0 \), but it is not locally connected at any point of \( X_1 \). No arc in \( X \) connects a point of \( X_1 \) to a point of \( X_0 \) even though points of \( X_0 \) can be found in each neighborhood of each point of \( X_1 \).

(4) \textit{Remark}. While it is not immediately obvious, the following is true. Every connected, locally connected subset of \( \mathbb{R}^n \) is arcwise connected. Example 3 demonstrates that the converse to this statement is not true.

Suppose that \( X \) is a continuum and \( A \) is a closed subset of \( X \). A \textit{component} \( K \) of \( A \) is a connected subset of \( A \) which is not a proper subset of any other connected subset of \( A \). Each point of \( A \) is contained in a component of \( A \), though in some cases, the component might be just a single point.

(5) \textit{Example}. Suppose that \( D \) is a closed disk in \( \mathbb{R}^2 \) (i.e. a circle and its interior). Suppose that \( X \) denotes the topologist’s \( \sin(1/x) \) curve and \( M \) denotes the Cantor fan. What do the components of \( D \cap X \) or \( D \cap M \) look like, assuming the intersection is nonempty and it is not all of \( X \) or \( M \)? It depends, of course, on which \( D \) is considered, but assuming \((\frac{1}{2},1) \notin D \), each component of \( D \cap X \) or \( D \cap M \) is a point or an arc. Further, if we are considering \( D \cap X \) and the interior of \( D \), denoted \( D^0 \), contains a point of \( X_1 \), then \( D \cap X \) has countably infinitely many components, all but possibly one of which is an arc. (See fig. 6.)

If \((\frac{1}{2},1) \) is not in \( D \) and the open disk \( D^0 \) intersects \( M \), then all, except possibly one, of the components of \( D \cap M \) are arcs and there are
uncountably many components in $D \cap M$. On the other hand, if $(\frac{1}{2}, 1) \in D^0$, then $D \cap M$ is itself a continuum homeomorphic to $M$. In this case, $D \cap M$ has only one component.

Neither $X$ nor $M$ is an “indecomposable” continuum. But we are heading in the direction of constructing examples that are “indecomposable”, for indecomposable continua are neither arcwise connected or locally connected at any point. We consider then some simple examples of indecomposable continua in dynamical systems in $\mathbb{R}^2$.

(6) Example. The invariant set of the Smale horseshoe map. This is the famous Smale horseshoe map. The construction begins as follows: Consider the stadium-shaped region called $D$ in fig. 7a. The set $D$ consists of a rectangle $R$ with interior, and two semicircles $A$ and $B$ (interiors included), that are sewn onto the shorter sides of $R$. Now $D \subseteq \mathbb{R}^2$ and the homeomorphism $F$ on $\mathbb{R}^2$ maps $D$ into itself as pictured in fig. 7b. Think of $F$ having the following effect on $D$: the map $F$ shrinks $D$ vertically, stretches $D$ horizontally, and then folds $D$ once and places the acted-upon $D$ back into itself so that $F(A)$ and $F(B)$ are in the interior of $A$ and $F(R)$ is in the interior of $D$.

Since $F(D) \subseteq D$, we have $F^2(D) = F(F(D)) \subseteq F(D)$. The set $F^2(D)$ is pictured in $F(D)$ in fig. 8. This process may be continued: $D \supseteq F(D) \supseteq F^2(D) \supseteq \ldots$. Since each $F^n(D)$ is a continuum, the sequence $D, F(D), F^2(D), \ldots$ is a sequence of nested continua. An elementary theorem from topology tells us that any intersection of nested continua is itself a nonempty continuum. It follows that $K = \cap_{n=1}^{\infty} F^n(D)$ is a continuum.

Now another theorem from topology states that any nested intersection of compact sets is nonempty and compact. Since $K \cap A$ can be written $\cap_{n=1}^{\infty} [F^n(D) \cap A]$ and $K \cap B$ can be written $\cap_{n=1}^{\infty} [F^n(D) \cap B]$ we have that $K \cap A$ is nonempty and $K \cap B$ is nonempty. Thus, $K$ contains more than one point. (A continuum that contains more than one point is said to be nondegenerate.) Note that if $x \in \mathbb{R}^2$ and there is some positive integer $n$ such that $F^n(x) \in D$, then the sequence $x, F(x), F^2(x), \ldots$ must be getting closer and closer to $K$. In other words, $K$ is the global attractor for $D$ in the sense that all initial points are attracted to $K$. $K$ is also an indecomposable continuum.

Precisely, a continuum $X$ is decomposable if it can be written as the union of two proper subcontinua $H$ and $K$. The sets $H$ and $K$ will overlap. A continuum that is not decomposable is indecomposable. The most commonly encountered continua are decomposable. For example, the interval $[0, 1]$ is the union of the two proper subcontinua $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Giving a rigorous proof that $K$ is indecomposable is rather tedious. We in-
stead attempt to give the reader an intuitive idea of what this all means. First, it may help to think of $K$ in another way. A set which is topologically equivalent to $K$ is called the Knaster bucket handle, usually denoted $K_2$, and it may be described as follows. (See fig. 9 for a sketch.) Suppose $C$ denotes the Cantor middle-thirds set sitting on the unit interval $[0, 1] \times \{0\}$ in the plane. (The “middle-thirds Cantor set” $C$ is the closure of the set of numbers in $[0, 1]$ which when written in base 3 (with digits 0, 1, and 2) in fact use only 0’s and 2’s, but no 1’s. That is, $x$ is in $C$ if its “decimal expansion” in base 3 contains no 1.) Connect the points of $C$ with semicircles as follows: (1) For each pair $p, q$ of points of $C$ such that $p$ and $q$ are equidistant from $(\frac{1}{2}, 0)$, connect $p$ and $q$ with a semicircle sitting above the $x$-axis. (2) For each pair $p, q$ of points of $C$ equidistant from $(\frac{1}{3}, 0)$ (the midpoint of $(\frac{2}{3}, 0)$ and $(1, 0)$), connect $p$ and $q$ with a semicircle that extends below the $x$-axis. (3) For each pair $p, q$ of points equidistant from the midpoint $(\frac{5}{3n}, 0)$ of $(\frac{2}{3}, 0)$ and $(\frac{1}{3}, 0)$, connect $p$ and $q$ with a semicircle that extends below the $x$-axis. Continue this process. (The $n$th step will consist of connecting each pair $p, q$ of points equidistant from the midpoint $(5/3^n, 0)$ of the line from $(2/3^n, 0)$ to $(1/3^n-1, 0)$. The points $p$ and $q$ are to be connected with a semicircle extending below the $x$-axis.) (We emphasize that although we give no proof here that $K$ and $K_2$ are topologically equivalent, comparing both constructions should at least convince the reader of the plausibility of this equivalence.)

Now suppose $S$ is a small closed disk in the plane. If the interior of $S$ intersects $K_2$, then $S \cap K_2$ consists of an uncountable collection of arcs (and possibly a couple of points on the boundary). If $S$ intersects the interior of the Cantor fan $M$, and $(\frac{1}{4}, 1)$ is not in $S$, then $S \cap M$ also is an uncountable collection of arcs (with possibly a couple of points on the boundary). Locally then, for the most part, $M$ and $K_2$ are the same topologically. However, there is one way in which $M$ and $K_2$ are very different, and that is the fact that $M$ has the vertex point $(\frac{1}{4}, 1)$ at which the whole continuum is connected. $K_2$ possesses no such point. In fact, a stronger statement may be made: Suppose $T$ is a closed set in $\mathbb{R}^2$ that contains a closed disk $S$, and that $K_2$ intersects the interior of $S$, but $K_2$ is not contained in $T$. Then $K_2 \cap T$ has uncountably many components. (In this case, each component is either a point or an arc.) This uncountable component property is the one that makes $K_2$ indecomposable and $M$ decomposable. Again, no matter where the interior of $S$ intersects $K_2$, as long as $K_2$ is not a subset of $T$, it follows that $K_2 \cap T$ has uncountably many “pieces”. On the other hand, there is one point of $M$ where this sort of property does not hold, namely at the vertex point $(\frac{1}{4}, 1)$.

By the beginning of this century mathematicians realized that just because an open bounded set in $\mathbb{R}^2$ is homeomorphic to an open disk, that does not mean that the closure of that set is homeomorphic to a closed disk. (This is the case with the blue, red, and green water regions in the Lake of Wada example after the canal construction.) It is known that any open set in the plane that is simply connected (has no “holes” and is connected) is homeomorphic to an open disk. It follows that if a set $U$ is the infinite union of an infinite number of sets that are nested ($U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots$) when the sets $U_i$ are homeomorphic to disks, then $U$ is homeomorphic to an open disk.

By 1932, Birkhoff [8] discovered his “remarkable curve,” which is actually a strange set and not an arc or a topological circle. He constructed
a map on an annulus with an unusual invariant set $G$. His set $G$ is the boundary set for an open set $G_1$ which contains one boundary circle, and is also the boundary set for an open set $G_2$ which contains the other boundary circle.

A point $p$ in $G$ is said to be accessible from $G_1$ if there exists an arc $A$ which has $p$ as an endpoint and has the property that the entire arc except for $p$ is a subset of $G_1$. We can define points in $G$ to be accessible from $G_2$ similarly. (Not all points in $G$ are accessible for $G_1$ or $G_2$.) In fact, “most” of the points in Birkhoff’s set $G$ are not accessible from either $G_1$ or $G_2$.) It is possible to talk about the rotation number of the points in $G$ accessible from $G_1$, and also the rotation number of the points in $G$ accessible from $G_2$. (We will not define this term precisely here. Loosely, what it measures is the average rotation of accessible points on $G$.) What is unusual about Birkhoff’s curve is the fact that the rotation number for points accessible from $G_1$ is different from the rotation number for points accessible from $G_2$. (This cannot happen on a circle or even anything “close” to being a circle, topologically.) In 1934, Charpentier [11], who was working as a postdoctoral fellow with Birkhoff, proved that Birkhoff’s curve is an indecomposable continuum.

A continuum in the plane is said to separate the plane if $\mathbb{R}^2 - X$ consists of two disjoint, nonempty open sets. (Circles separate the plane, as does the Lakes of Wada continuum $W$, which separates the plane into red, blue, and green regions. The Cantor fan $M$, topologist’s $\sin(1/x)$ curve $L$, and horseshoe attractor $K$ do not separate the plane.) Cartwright and Littlewood studied the forced van der Pol equations in a pair of papers that appeared in 1945 and 1951, respectively [9, 10]. They found that, at certain parameter values, an associated Poincaré homeomorphism, a Poincaré return map, admits a certain invariant, plane-separating continuum. Cartwright and Littlewood conjectured that this continuum contains an indecomposable continuum. In 1987, Barge and Gillette [2] proved that this continuum of Cartwright and Littlewood not only contains an indecomposable continuum, but is an indecomposable continuum.

In the 1960’s Smale [17, 18] invented his famous horseshoe maps (in order to gain a better understanding of the Cartwright and Littlewood results on the van der Pol equation). As discussed earlier, the horseshoe has an indecomposable continuum. Robert Williams, trained as a topologist, was also working in dynamics by this time. His work has a very topological flavor to it. He introduced important tools into dynamics called inverse limits and branched manifolds. (See refs. 19–21, for example.) These tools led him to indecomposable continua, one notable example being a class of continua known as the solenoids. Solenoids occur in differential equations. The limiting invariant set that occurs as a parameter is varied through a cascade a period doublings is a solenoid. For maps the limit set is a Cantor set, but for a flow it is a solenoid.

In recent years the greatest advances in the study of indecomposable continua in dynamical systems have been by Marcy Barge and his collaborators. Barge began his career as an applied mathematician studying differential equations, but began collaborating with a topologist, J. Martin. Barge and Martin subsequently coauthored several papers on the connection between inverse limits and indecomposable continua in dynamical systems [3–6].

5. Conclusion

We have seen that indecomposable continuas, sets with complicated topology, often occur with complicated dynamics. It is time for a word of warning: even more complicated topology is possible, even in very nice dynamical systems. In the examples we have examined, the proper subcontinua are either arcs or points. It is possible for a plane continuum that contains more than one point to fail to contain any arcs at all. Indeed, a much stronger statement may be made: “most”
continua in the plane do not contain arcs. (“Most” has a precise mathematical meaning which is too technical to go into here. See ref. [7].)

A continuum is said to be hereditarily indecomposable if each of its subcontinua is indecomposable. These continua do exist, they form most continua in $\mathbb{R}^2$, and they do not contain arcs. (Since arcs are decomposable, a hereditarily indecomposable continuum cannot contain any.) These extremely complicated sets are not the subject of this paper: we mention them because (1) they do occur in nice dynamical systems (see ref. [13]), and (2) it is not safe to assume that every indecomposable continuum contains “simple” subcontinua.

This collaboration was begun as a result of a series of lectures by the first author for the Dynamics Group, R44, at the Naval Surface Warfare Center in Silver Spring, Maryland.

Appendix

The pendulum example is difficult or impossible to analyze rigorously, so to establish that the claims are plausible, we present an idealization in which we can prove that (1) the basins of attraction (there are three) possess the Wada property; (2) there is a saddle fixed point whose stable manifold is dense in the common boundary of the basins and whose unstable manifold intersects all three basins. The proofs are fairly technical, and we sketch them to indicate the kinds of mathematical ideas needed for a detailed proof.

Consider the space which is the product of the circle $S^1$ and the real line $\mathbb{R}$, $S^1 \times \mathbb{R}$. We define a diffeomorphism $W$ on $S^1 \times \mathbb{R}$. The map $W$ is the composition of three simpler maps $s$, $\sigma$, and $\alpha$. For $(x, y) \in S^1 \times \mathbb{R}$, $s(x, y) = (x, \frac{1}{128} y + \frac{171}{128} \sin 2 \pi x)$. Note that $s$ fixes the sine curve, the points $(x, y)$ satisfying $y = \sin 2 \pi x$, and $s$ maps the annulus $A = S^1 \times [-2, 2]$ into a band of height $\frac{1}{4}$ about the sine curve. The map $s$ ripples and contracts $A$.

The map $\sigma$ is a shear. For $(x, y)$ in $S^1 \times \mathbb{R}$, define $\sigma(x, y) = (y + (x + y) \text{mod} 1, y)$. The composite map $\sigma \circ s$ contracts, ripples, and shears $A$. It also maps $A$ into itself and both $s$ and $\sigma$ have fixed points $(0,0), (\frac{1}{2} \pi, 1), (\frac{1}{4} \pi, 0), (\frac{1}{4} \pi, -1)$.

Write $(\frac{1}{4}, 1) = p_R, (\frac{1}{2}, 0) = p_B, and (\frac{3}{4}, 1) = p_G$. We define the third map $\alpha$ as follows. Suppose $(x, y)$ is a point of $S^1 \times \mathbb{R}$ which is at a distance $\frac{1}{4}$ or more from each of $p_R$, $p_B$, and $p_G$. Then $\alpha(x, y) = (x, y)$. If $(x, y)$ is within $\frac{1}{4}$ of $p_R$, $p_B$, or $p_G$, then it is within $\frac{1}{4}$ of only one of these points. If $(x, y)$ is inside the circle of radius $\frac{1}{4}$ centered at $p_R$, then it lies on exactly one radial line $L$ of the circle. The map $\alpha$ maps each radial line $L$ to itself, leaving fixed $p_R$ and the point on both the radial line $L$ and circle. If a point is on $L$ at a distance $0 < d \leq \frac{3}{16}$ from $p_R$, then that point maps to the unique point on $L$ at a distance $d(d + \frac{1}{16})$ from $p_R$. We do not specify further what happens to points inside the circle at a distance greater than $\frac{3}{16}$ from $p_R$, except to require that the resulting $\alpha$ be continuously differentiable. (This can be achieved in many ways.) Inside circles of radius $\frac{1}{4}$ about $p_B$ and $p_G$, we define $\alpha$ analogously.

Now $\alpha(A) = A$ and $\sigma \circ s(A) \subseteq A$. Therefore, $W(A) = \sigma \circ s(A) \subseteq A$, which implies $W^{n+1}(A)$ is a nested collection of continua, and $\mathcal{W} = \bigcap_{n=0}^{\infty} W^n(A)$ is a continuum. Further, since for each $n$, the set $W^n(A)$ separates $S^1 \times \mathbb{R}$ into two disjoint, open sets, each homeomorphic to $S^1 \times \mathbb{R}$, $\mathcal{W}$ also separates $S^1 \times \mathbb{R}$ into two disjoint open sets, each homeomorphic to $S^1 \times \mathbb{R}$. Let $U$ denote those points of $S^1 \times \mathbb{R}$ which lie in the open set of points above $\mathcal{W}$ and $V$ denote those points of $S^1 \times \mathbb{R}$ which lie in the open set below $\mathcal{W}$.

Lemma 1. The points $p_R$, $p_B$ and $p_G$ are attracting fixed points for $W$. This strip $[\frac{3}{16}, \frac{7}{16}] \times [-2, 2]$ is a subset of the basin of attraction of $p_R$, the strip $[\frac{5}{16}, \frac{9}{16}] \times [-2, 2]$ is a subset of the basin of attraction of $p_B$, and the strip $[\frac{11}{16}, \frac{15}{16}] \times [-2, 2]$ is
a subset of the basin of attraction of $p_G$. The fixed point $(0,0)$ is a saddle point for $W$.

The proof is omitted since it follows from direct calculations.

Let $U_R$, which we call the “red basin,” $U_B$, and $U_G$, (the “blue” and “green” basins) denote the basins of attraction of $p_R$, $p_B$ and $p_G$, respectively. Each of these sets can be shown to be homeomorphic to an open disk.

**Lemma 2.** Suppose that $p$ is a point $S^1 \times [0,1]$ that is in the boundary of one of $U_R$, $U_B$, and $U_G$. Then $p$ is in the boundary of each $U_R$, $U_B$, and $U_G$, i.e., $\partial U_R \cup \partial U_B \cup \partial U_G = \partial U_R \cap \partial U_B \cap \partial U_G$, where $\partial$ denotes the boundary.

**Outline of proof.** Let $A^0$ be the interior of $A$. The component $C_R$ of $U_R \cap A^0$ that contains $[\frac{1}{2}, \frac{3}{2}] \times (-2,2)$ is homeomorphic to the interior of an open disk (that is, it is connected and has no holes in it), and its closure intersects both boundaries of $A$. Examination of $W(A)$ reveals that this component $C_R$ intersects $W(A)$ in at least four components that run from the top of $W(A)$ to the bottom of $W(A)$. Thus, $W^{-1}(C_R \cap W(A))$ has at least four components that stretch between both boundaries of $A$. Similar statements can be made about the corresponding components $C_B$ and $C_G$ of $U_B \cap A^0$ and $U_G \cap A^0$, respectively, except that $C_B \cap W(A)$ has at least three components that run from between both boundaries of $W(A)$.

Further, the components of the basins in $A$ discussed so far alternate in a predictable, orderly way, and all run between both boundaries of $A$. If a connected open subset of $A$ extends from the top of $A$ to the bottom of $A$, we call it a basin strip. We look at the inverse image of the basin strips, and notice that each inverse image of a basin strip contains several basin strips in $A$. Continuing the process generates ever more basin strips, all disjoint. There are infinitely many basin strips and between any two, there are red, blue, and green basin strips, that is, basin strips in the basins of $p_R$, $p_B$, and $p_G$. Showing that each boundary in $A$ of each strip is actually a limit of strips of all colors requires some work, but eventually we reach our conclusion. □

A continuum $X$ is said to be arclike if for each $\epsilon > 0$, there is a map $f_\epsilon$ from $X$ onto $[0,1]$ such that for each $x \in [0,1]$, $\text{diam} f_\epsilon^{-1}(x) < \epsilon$. A continuum $X$ is circlelike if for each $\epsilon > 0$, there is a map $g_\epsilon$ from $X$ onto the circle $S^1$ such that for each $x \in S^1$, $\text{diam} g_\epsilon^{-1}(x) < \epsilon$. Roughly, an arclike (circlelike) continuum is one that can be approximated by arcs (circles). Arclike and circlelike continua have been widely studied. Arcs, the topologist’s $\sin(1/x)$-curve, and the Knaster continuum are examples of arclike continua. The circle $S^1$ and the solenoids are circlelike. As we see next, $\hat{W}$ is also circlelike (though it is not homeomorphic to a circle).

If $x$ is a point in the continuum $X$, then the composant of $x$ in $X$ is the union of all proper subcontinua of $X$ that contain $x$. If $X$ is an indecomposable continuum, then the composants of $X$ form an uncountably infinite collection of mutually disjoint, dense sets.

**Lemma 3.** The global attractor $\hat{W}$ is circlelike and indecomposable with

1. each composant having the property that it is the one-to-one image of the line $R^1$, or the one-to-one image of the ray $[0, \infty)$ of nonnegative numbers, and

2. the composant that contains $(0,0)$ is the unstable manifold of $(0,0)$.

**Outline of proof.** Note that $\hat{W}$ is the nested intersection of annuli. We leave out some details here, but eventually we can conclude that $\hat{W}$ is circlelike. No proper subcontinuum of $\hat{W}$ contains two of the points in $\{(0,0), p_R, p_B, p_G\}$, so $\hat{W}$ is indecomposable. Properties (1) and (2) then follow easily. □

The set $\hat{W}$ is circlelike in one other way: it contains no triod, that is, it has no subset homeomorphic to the letter $Y$. 

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**Theorem 4.** The unstable manifold of (0, 0) crosses all three of the basins $U_R$, $U_B$, and $U_G$. The stable manifold of (0, 0) intersects both boundaries of $A$ and is dense in the common basin boundary. Every point in $S^1 \times \mathbb{R}$ is either in $U_R$, $U_B$, or $U_G$, or is in the basin boundary.

**Outline of proof.** The first statement is obvious. The stable manifold must cross at least one boundary of some $W^n(A)$, since $W$ contains no triods and cannot contain the stable manifold. Symmetry considerations mean that the stable manifold crosses both boundaries of $W^n(A)$, which in turns means that the stable manifold crosses both boundaries of $A$.

Suppose $\theta$ is an open disk that intersects the basin boundary. There is an arc $P$ in $\theta$ that intersects all three basins. Since $P$ is compact, there is an $n$ such that $W^n(P) \subseteq A$ and $W^n(P)$ comes within $\frac{1}{16}$ unit of each of $P_R$, $P_B$, and $P_G$. Then $W^{n+1}(P)$ intersects the stable manifold. Therefore, $\theta$ intersects the stable manifold, and the stable manifold is dense in the basin boundary. From the proof of proposition 1, it follows that the stable manifold is in the boundary of $U_R$ from “both” sides. Likewise, it is in the boundary of $U_B$ from “both” sides, and in the boundary of $U_G$ from “both” sides. The last statement follows. □

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