A PROCEDURE FOR FINDING NUMERICAL TRAJECTORIES ON CHAOTIC SADDLES*

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Examples are common in dynamical systems in which there are regions containing chaotic sets that are not attractors. If almost every trajectory eventually leaves some region, but the region contains a chaotic set, then typical trajectories will behave chaotically for a while and then will leave the region, and so we will observe chaotic transients. The main objective that will be addressed is the "Dynamic Restraint Problem": Given a region that contains a chaotic set but does not contain an attractor, find a chaotic trajectory numerically that remains in the region for an arbitrarily long period of time. Systems with horseshoes have such regions as do systems with fractal basin boundaries, as does the Hénon map for suitably chosen parameters.

We present a numerical technique for finding trajectories which will stay in such chaotic sets for arbitrarily long periods of time, and it leads to a "saddle straddle trajectory". The method is called the "PIM triple procedure" since it is based on so-called PIM triples. A PIM (Proper Interior Maximum) triple is three points (\(a, c, b\)) in a straight line segment such that the interior point \(c\) (i.e. \(c\) is between \(a\) and \(b\)) has the maximum escape time, \(t_{\text{max}}\), that is, its escape time from the region is greater than the escape time of both \(a\) and \(b\). "Proper" means the segment from \(a\) to \(b\) is smaller than a previously obtained segment. We show rigorously that the PIM triple procedure works in ideal situations. We find it works well even in less than ideal cases. This procedure can also be used for the computation of Lyapunov exponents.

Furthermore, the "accessible PIM triple procedure" (a refined PIM triple procedure for finding accessible trajectories on the chaotic saddle) will also be discussed.

1. Introduction

Studying dynamical systems, one often observes chaotic transient behavior, apparently due to the presence of horseshoes. For example, for suitably chosen parameter values, the Hénon map has an attracting period orbit with period 5, and one observes that the duration of chaotic transient behavior of many trajectories is rather short before they settle down on the period 5 attractor. Other famous examples are: the dynamics of a bouncing ball, the forced damped pendulum, and the Lorenz equations.

Let \(F\) be a differentiable, invertible map from the plane to itself, such that the derivatives of \(F\) and its inverse are continuous. A region \(R\) is a closed and bounded set in the plane such that \(R\) is the closure of its interior. We say a region \(R\) is a restraining region if it contains no attractor. We investigate trajectories that remain in such a region \(R\). For example, the horseshoe is usually pictured mapping a rectangle to a horseshoe shape; the box is a restraining region.
Let $R$ be a restraining region for $F$. We assume throughout that almost every initial condition is on a trajectory that eventually leaves $R$. The **stable set** $S(R)$ is the set of points (in $R$) which stay in $R$ for all time under forward iteration of $F$; the **unstable set** $U(R)$ is the set of points which stay in $R$ for all time under backward iteration of $F$. The set of points in $R$ which stay in $R$ for all time under forward and backward iteration of $F$ is called the **invariant set** $\text{Inv}(R)$ of $F$ in $R$, that is, $\text{Inv}(R) = S(R) \cap U(R)$. We call $\text{Inv}(R)$ a **chaotic saddle** when it includes a Cantor set. We assume throughout that $\text{Inv}(R)$ is nonempty.

If almost every point eventually leaves a region (possibly returning to this region later) then usually the mean time for leaving this region is a few iterates. The great majority of the trajectories of the horseshoe map will leave the region after a few iterates. We are looking for trajectories that stay in the region as long as we wish to compute them. The main question that we would like to address is:

**The dynamic restraint problem.** Find a chaotic trajectory numerically that remains in a specified region for an arbitrarily long period of time.

Assume that there exists an invariant Cantor set and assume that there is a trajectory that is dense in this Cantor set. We want a procedure that finds such a trajectory. We would like to emphasize that the above question leads to the following "dual" problem:

**The static restraint problem.** Find an initial point whose trajectory is chaotic and stays in a specified region for an arbitrarily long period of time.

In both above problems, the trajectories should be chaotic, that is, irregular, nonperiodic, and hopefully dense in the chaotic set.

We will describe a procedure for following trajectories in specified regions for dynamical systems in the plane. There are situations for which the procedure cannot be carried out, but it does work in practice in many cases including those described below. We will restrict ourselves to some examples of lower dimensional systems such as the horseshoe map, the Hénon map, and the Lorenz equations. We investigate trajectories that remain in a given restraining region $R$: for the horseshoe the region $R$ is the box $B$ as in fig. 2. Another example, which has been studied by Kantz and Grassberger [8], is when the Hénon map has a stable 5-cycle, the region $R$ is a box from which 5 small disks have been deleted, see fig. 1.

The **escape time** $T(x)$ of a point $x$ in the restraining region $R$ under the map $F$ is the minimum value $n$ with the property that $F^n(x)$ is not in $R$; the escape time $T(x)$ is infinity if $F^n(x)$ is in $R$ for all $n$.

We first consider the intensively studied horseshoe map $F$, which is defined on a neighborhood of a box $B$. The intersection $F(B) \cap B$ consists of two vertical strips, say $V_1$ and $V_2$. Let $H_1$ and $H_2$ be the horizontal strips in $B$ (stretching the full width of $B$) such that $F(H_1) = V_1$ and $F(H_2) = V_2$; see fig. 2. It is well known, see e.g. Guckenheimer and Holmes [6] and section 4, that under reasonable assumptions, the stable set $S(B)$ of $F$ w.r.t. $B$ is a Cantor set of horizontal curves (stable segments), and the unstable set $U(B)$ of $F$ w.r.t. $B$ is a Cantor set of vertical curves (unstable seg-

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**Fig. 1.** Restraining region is a box from which 5 small disks have been deleted.
The intersection $C$ of the stable set with the unstable set in $B$ is a chaotic saddle. Note that all the points on the chaotic saddle $C$ stay in the box for all time under all forward and all backward iterates of the map $F$. The stable (resp. unstable) set will also be called the stable (resp. unstable) set of the chaotic saddle.

In this paper we restrict our attention to cases in which there is only one positive Lyapunov exponent; that is, the saddle is one-dimensionally unstable.

For the example of the horseshoe map, the graph in fig. 3 shows the escape time function $T$ on an unstable segment.

Fig. 3 suggests the following facts:

Fact 1. $T(x)$ is infinite on a Cantor set of lines.
Fact 2. $T(x)$ increases monotonically to infinity as the Cantor set is approached.
Fact 3. If $a$, $c$, and $b$ are three points on an unstable segment such that $c$ is between $a$ and $b$ and, $T(c) > T(a)$ and $T(c) > T(b)$ then the straight line segment $[a, b]$ from $a$ to $b$ intersects the stable set of the chaotic saddle.

These facts play a crucial role in developing the PIM triple procedure. In evaluating the significance of PIM triples, the reader may wish to compare it with the PIM effect [F. Hoyle, Element 79, The New American Library, Inc., New York City 1967, 25-31.]

The organization of the paper is as follows. In section 2 the PIM triple procedure will be described. Then, in section 3 we present some examples in which the PIM triple procedure has been applied, and we discuss the extent to which the procedure might be used. The PIM triple procedure will be established for a class of horseshoe maps in section 4 and the appendix. Finally, the “accessible PIM triple procedure” is discussed in section 5, and we establish this procedure for the same class of horseshoe maps.

The proof of the PIM triple procedure (and the accessible PIM triple procedure) for horseshoes can be extended in one and two dimensions to all hyperbolic maps, though then the proof is considerably longer.

2. The PIM triple procedure

Let $J$ be a segment that crosses the stable set in the restraining region $R$. The notation $(a, c, b)$ denotes a collection of three points in $J$, with $c$ lying between $a$ and $b$. From now on, let $a$, $b$, and $c$ be three distinct points on $J$ where the ordering on $J$ is such that $a < c < b$. The choice of which direction in the segment $J$ is “increasing”
is arbitrary since we use the ordering only to discuss when a point is between two others. For points \( x \) and \( y \) in \( J \) we write \([x, y]\) for the subsegment of \( J \) joining \( x \) and \( y \), and \(|x - y|\) for the distance between \( x \) and \( y \) along \( J \).

The triple \((a, c, b)\) is called an Interior Maximum triple if \( T(c) > \max\{T(a), T(b)\}\); and \((a, c, b)\) is called a Proper Interior Maximum triple on \( J \), abbreviated PIM triple, if \((a, c, b)\) is an Interior Maximum triple and \([a, b]\) is a proper subset of \( J \). We will call the interval \([a, b]\) a PIM triple interval.

For each \( \varepsilon > 0 \), an \( \varepsilon \)-refinement of the two points \( a \) and \( b \) w.r.t. \( J \) is a finite set of points \( \{g_j\} \) in \( J \) with \( a = g_0 < g_1 < \cdots < g_N = b \) such that \( |g_k - g_{k+1}| \leq \varepsilon \cdot |a - b| \), and an \( \varepsilon \)-refinement of the triple \((a, c, b)\) w.r.t. \( J \) is an \( \varepsilon \)-refinement of the two points \( a \) and \( b \) as above so that \( c = g_k \) for some \( k, 1 \leq k \leq N \). As a special case of an \( \varepsilon \)-refinement of two points \( a \) and \( b \) the reader should keep in mind the partition of the interval \([a, b]\) into \( N \) equal subintervals for some integer \( N \geq 1/\varepsilon \). We will illustrate this with an example.

**Example.** The Hénon map \( F(x, y) = (1.45 - x^2 + 0.2y, x) \) has an attracting period 5 orbit (see section 3 for a detailed account). Let \( B = \{(x, y): -2 \leq x, y \leq 2\} \) and let \( B_1 \) be the open ball of radius 0.005 centered at one of the points of the attracting periodic orbit. We choose \( R = B \setminus B_1 \), the box \( B \) from which the open ball \( B_1 \) is deleted. Notice that points which are attracted to the period 5 orbit will leave \( R \). We choose an interval to start the numerical procedure, namely, the horizontal line extending from the left side to the right side of the box \( B \) with \( y = 1 \). Now we select \( N = 30 \), that is, we divide the interval into 30 subintervals of the same length. Note that the grid of end points of these 30 smaller intervals is an \( \varepsilon \)-refinement with uniform spacing. The escape time of these grid points are presented in fig. 4.

In fig. 4 we observe that there are 10 triples of 3 consecutive points that constitute PIM triples. Our experience so far is that typically there are several PIM triples consisting of three consecutive grid points by selecting \( N = 30 \). In this example the maximal escape time is 180, and there is precisely one point (the point numbered 23) having this escape time. The three consecutive points 22, 23, and 24 constitute a PIM triple.

Lorenz reported the phenomenon of irregular escape times in [9] while investigating the structure of an attractor for a four-dimensional system of differential equations. Write \( S \) for some fixed ball which contains the attractor, and \( A \) for the attractor. For each point in \( S \), but not on the attractor \( A \), one can determine the escape time from \( S \) by integrating backward. As Lorenz noticed, large values for the escape time presumably imply proximity to the attractor. Supposing that the attractor was the product of a 3-dimensional continuum and a Cantor set, he chose a line segment that appeared to cross the sheets of the attractor and studied escape times along that segment. Lorenz noticed that if the attractor sheets were continua, then the maximum values of the escape time should
increase without limit with increased resolution along the segment. The computations of the escape time yielded values which varied erratically, even with very high resolution. Furthermore whenever he noticed an exceptionally high value of the time escape for some point along the line segment, he always succeeded in finding a still higher value of a point nearby, by increasing the resolution.

The idea of the procedure to “Find a point whose trajectory is chaotic and stays in the restraining region” is the following. Let $\epsilon > 0$ be small. Given a PIM triple $(a_0, c_0, b_0)$, choose some $\epsilon$-refinement of $(a_0, c_0, b_0)$, select 3 not necessarily consecutive points from this refinement giving a new PIM triple $(a_1, c_1, b_1)$. Why we may expect there to be such a PIM triple will be explained below. Note that, according to the definition of PIM triple, $|b_1 - a_1| < |b_0 - a_0|$. Choose some $\epsilon$-refinement of the PIM triple $(a_1, c_1, b_1)$, select in this refinement again a new PIM triple $(a_2, c_2, b_2)$; and repeat the procedure. At the $n$th refinement stage one chooses an $\epsilon$-refinement of a PIM triple $(a_n, c_n, b_n)$ resulting in a new PIM triple $(a_{n+1}, c_{n+1}, b_{n+1})$ which we may assume satisfies $|b_{n+1} - a_{n+1}| \leq (1 - \epsilon) \cdot |b_n - a_n|$. Thus the nested sequence of the intervals $\{[a_n, b_n]\}_{n \geq 0}$ converges to a point which we will call a PIM limit point. It follows now that the escape time of $a_n$ goes to $\infty$ and the trajectory of the PIM limit point stays in the restraining region. This “static” problem’s solution is not directly implementable on a computer because of the limitations of numerical precision, but it leads to the solution of the dynamic restraint problem as discussed below.

Now we will return to the “dynamic” question addressed in the introduction, namely, how can you numerically follow a trajectory on a chaotic saddle for an arbitrarily long period of time?

A line segment $[a, b]$ straddles the “stable set” of a point $P$ (i.e., the stable manifold of $P$, the set of points whose trajectories approach asymptotically the trajectory of $P$) if the segment crosses this set; see fig. 5. In the cases we study $P$ will be replaced by a chaotic saddle and $[a, b]$ will straddle some piece of its stable set, that is, the set of points attracted to the chaotic saddle. Furthermore, in practice $[a, b]$ will be extremely close to the chaotic saddle.

The numerical procedure goes as follows: (1) Choose (with some experimenting) an interval $L$. (2) Apply the PIM triple procedure repeatedly finding a sequence of successively smaller PIM triple intervals until the length of the PIM triple interval is less than some small distance that we will refer to as $10^{-8}$; call this interval $I_0$. (3) Iterate the interval $I_0$ forward: let $I$ denote the interval whose end points are the images under the map of the end points of $I_0$; let $I_1 = I$ unless the length of $I$ exceeds $10^{-8}$, in which case one applies the PIM triple procedure to decrease its length until it is less than $10^{-8}$. Then the new PIM triple interval is denoted $I_1$. These steps are then applied to $I_1$ to obtain $I_2$, etc. We will say we “iterate the interval” when we apply the map (or differential equation solver) once to $a$ and $b$, the end points of the interval.

We thus obtain a trajectory of tiny intervals $I_n$, with $I_{n+1} \subset F(I_n)$. Of course, this is not exactly true because $F(I_n)$ is curved, but the curvature is very slight, typically deviating from a straight line by about $10^{-16}$, which is on the order of the typical round-off error.
If we select any point $x_n$ from $I_n$, perhaps the midpoint, we know that

$$|x_{n+1} - F(x_n)|$$

is small, typically of the order of $10^{-8}$. Since computers can never be expected to produce true chaotic trajectories without noise, we may say $\{x_n\}_{n \geq 0}$ is a numerical trajectory with numerical precision of the order of $10^{-8}$. We call the sequence of intervals $\{I_n\}_{n \geq 0}$ a saddle straddle trajectory because the interval straddles a piece of the stable set (of a chaotic saddle) as a child straddles a stream with one foot on each side, and this trajectory approximates the chaotic set (which is a chaotic saddle). Despite the complexity of the construction, we may view $x_{n+1}$ as the "iterate" of $x_n$ (with precision $10^{-8}$).

3. Examples

We first present some applications of the PIM triple procedure. The PIM triple procedure could also be used for straddling the basin boundary of two attractors, but for obtaining such a basin straddle orbit a much simpler "bisection" procedure introduced in Grebogi et al. [5] can be used. In all the examples below, the pictures were obtained by using the Dynamics Program [12] using $N = 30$.

3.1. Hénon map

Let the diffeomorphism $F$ acting on the plane be given by

$$F(x, y) = (A - x^2 + M \cdot y, x).$$

The map $F$ is equivalent under a change of variables to the Hénon map $(1 - A \cdot X^2 + Y, M \cdot X)$.

For a first example, we choose the parameters $A = 2$ and $M = 0.3$. This choice of the parameters does not satisfy the conditions, stated by Devaney and Nitecki [3], that $F$ is a topological horseshoe. We select the restraining region $R$ to be the box $\{(x, y): -3 \leq x \leq 3, -3 \leq y \leq 3\}$. We start the numerical procedure with the horizontal line extending from the left side to the right side of the region $R$ with $y = 1$. By using the "PIM triple procedure" we obtain a trajectory consisting of more than 20000 points (actually tiny intervals). The result is presented in fig. 6(a).

For a second example, we select the parameter values $A = 1.45, M = 0.2$. This example has previously been considered by Kantz and Grassberger [8]. The map $F$ has an attracting cycle with period 5, and the box $B = \{(x, y): -2 \leq x \leq 2, -2 \leq y \leq 2\}$ contains a chaotic saddle in its interior. Kantz and Grassberger [8] gave the following numerical procedure for determining an invariant distribution on the chaotic saddle. They started with a certain number of initial points being uniformly distributed in the box. For each starting point, the first ten iterations were not plotted; and the iterations were stopped when both Lyapunov exponents (averaged over the last 15 iterations) were negative, and the last 30 points were not plotted. With this procedure, Kantz and Grassberger obtained for this example "104 chaotic trajectories (containing 10^5 points)", see [8] for details.

Notice that their procedures require discarding 40 points of each trajectory. Their procedure will work satisfactorily only if the trajectories tend to stay in the region for a long time. If half of the points exit on each iterate, they will discard roughly on the average 2^40 trajectories before any points are plotted. In contrast, our procedure results in a single trajectory whose statistics (such as Lyapunov exponents) can be computed directly. They obtain fragments of trajectories, so Lyapunov exponent computations must piece together results from these fragments. For systems with very long chaotic transients their procedure is preferable. Our procedure runs more quickly as
the mean exit time decreases; theirs requires long exit times.

We selected the restraining region $R$ to be the box $B$ with 1 ball deleted, the ball of radius 0.005 around one of the stable periodic points (in order not to slow down the computation we deleted 1 rather than 5 balls as mentioned earlier). Applying the "PIM triple procedure" results in one numerical trajectory, that has been presented in fig. 6(b). It appears that we can obtain as many iterates of our straddle trajectory as we want, hence the numerical trajectory will stay in the restraining region for an arbitrarily long period of time. Our fig. 6(b) is almost identical to the Kantz-Grassberger figure. In fact, their picture has a small extra piece which we believe is due to their discarding only 10 initial points of each trajectory.

For the computation of the Lyapunov exponent they used a method slightly different from the one for producing the picture and obtained $\lambda_1 = 0.42 \pm 0.002$. Note that the second exponent satisfies $\lambda_2 = \log 0.2 - \lambda_1$. Our PIM triple procedure gives virtually identical results for $\lambda_1$ for a trajectory consisting of $4 \times 10^5$ points.

3.2. The dynamics of a bouncing ball

Guckenheimer and Holmes [6] (see also the references therein) considered a two-dimensional map which provides a model for repeated impacts of a ball with a massive, sinusoidally vibrating table, yielding the following map:

$$f_{\alpha, \beta}(x, y) = (x + y (\mod 2\pi) - \pi, \alpha y + \beta \sin (x + y)).$$

They showed that the map $f_{\alpha, \beta}$ is a diffeomorphism.

The map $f_{\alpha, \beta}$ has a chaotic saddle for many values of $\alpha$ and $\beta$, see Guckenheimer and Holmes [6], section 5.2 for details. We select from these values $\alpha = 0.5$ and $\beta = 16$, and apply the PIM triple procedure for some different restraining re-
3.3. Pendulum

We consider the differential equation

\[ x''(t) + vx'(t) + \sin x(t) = f \cos(t). \]

We apply the PIM triple procedure with the parameter values \( v = 0.2 \) and \( f = 2 \). For these parameters, the time-2\( \pi \) map has two stable fixed points. It was already observed that there was transient behavior in these basins. We choose the restraining region to be the rectangle \((x, y): -\pi \leq x \leq \pi, -3 \leq y \leq 4\) from which two balls (of radius 0.05) centered at these attractors, have been
Fig. 7. Continued. (e) the attractor in the region $-\pi \leq x \leq \pi$, $-32 \leq y \leq 32$; the small box is the chosen restraining region; (f) numerical trajectory obtained by PIM triple procedure in the restraining region $0 \leq x \leq \pi$, $15 \leq y \leq 31$; (g) the attractor in the region $-\pi \leq x \leq \pi$, $-32 \leq y \leq 32$; the small box is the chosen restraining region; (h) numerical trajectory obtained by PIM triple procedure in the restraining region $0 \leq x \leq \pi$, $22 \leq y \leq 30$.

deleted. The result for the choice of the interval with end points $(-3, -3)$ and $(3, 4)$ is presented in fig. 8(a); the white coloured basin includes this trajectory, see fig. 8(b). We would like to point out that the segment from $(-3, 4)$ to $(3, -3)$ gives a numerical trajectory in the black coloured basin.

3.4. Lorenz equations

We consider the Lorenz equations

\[ \dot{x} = \sigma (x - y), \]
\[ \dot{y} = rx - y - xz, \]
\[ \dot{z} = xy - bz. \]
We apply the PIM triple procedure with the parameter values $r = 16$, $b = 8/3$, $\sigma = 10$. For these parameters there is no chaotic attractor, but there is transient chaos, and there are two attracting equilibria with $z = r - 1$. We choose the restraining region to be $\mathbb{R}^3$ minus unit balls centered at these attractors. The result is presented, plotted in the $(x, z)$ plane, in fig. 9. We expect that the straddle trajectory would eventually go to the unstable equilibrium point $(0, 0, 0)$, though we did not observe this tendency in our computer runs.

From the examples above, we have seen that the PIM triple procedure works quite well for different dynamical systems. For many dynamical systems in the plane with an invariant chaotic saddle $C$ in a restraining region $R$ we believe that the PIM triple procedure will result in saddle straddle trajectories that stay in the region $R$ for as long as the computation is continued.

It is important to ask if such straddle trajectories represent true trajectories of the system. In other words, does there exist a true trajectory of

Fig. 9. Numerical trajectory obtained by the PIM triple procedure for the Lorenz equations for the parameter values $r = 16$, $\sigma = 10$, and $b = 2.666667$. Blow-up of the little box in the restraining region in the upper right section: horizontal axis: $-1.18555 \leq x \leq 1.1582$; vertical axis: $2.293716 \leq z \leq 4.91447$. 

Fig. 8. Numerical trajectory obtained by the PIM triple procedure for the forced pendulum equation for the parameters $\nu = 0.2$, $f = 2$. 

Fig. 8. Numerical trajectory obtained by the PIM triple procedure for the forced pendulum equation for the parameters $\nu = 0.2$, $f = 2$. 

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It is important to ask if such straddle trajectories represent true trajectories of the system. In other words, does there exist a true trajectory of
the system that shadows (i.e., stays close) to the numerical trajectory obtained by the PIM triple procedure? When a map is sufficiently hyperbolic on the invariant set in question, Bowen [2] obtained a result saying that each noisy trajectory can be shadowed by a true trajectory if the perturbation is small; see [2] for the precise statement. We believe that just about every saddle straddle trajectory of a two-dimensional (non-hyperbolic) system with a chaotic saddle, obtained by the PIM triple procedure, can be shadowed by a true trajectory for as long as the saddle straddle trajectory can be computed. It should be noted however that the PIM triple procedure does fail in some cases where either the restraining region $R$ is not chosen appropriately or the trajectory reaches a point of the saddle set at which the dynamical system is not hyperbolic. Hammel et al. [7] report that for non-hyperbolic attractors (in particular the Hénon map) typical numerical trajectories can be shadowed by true trajectories for long periods of time but not forever, so we need not expect more from non-hyperbolic chaotic saddles. There were no failures in the cases reported in this paper.

In section 2 we have argued that when each interval is divided into 30 subintervals of equal length one may expect reasonable results for the PIM triple procedure in many cases. In order to give the reader an idea about the escape times of the points in some grids, and to give the reader the opportunity to compare these escape times with those presented in fig. 5, we have collected them in table I for $N = 100$.

Also we noticed in section 2 that the $e$-refinement in which $e = 1/30$ and the distance of two consecutive points is uniform has 10 PIM triples consisting of three consecutive points, and it has precisely one such “maximal” PIM triple, that is, for which the escape time assumes the maximal value of the escape time on the grid of points. In table II we have collected these kind of results for $2 < N < 50$; from this information, the reader might suspect that our choice for a grid of 31 points is reasonable.

<p>| Table I |
| Escape time on a grid. $n$: number of the grid point, $0 \leq n \leq 100$. $T(n)$: escape time of grid point $n$, $0 \leq n \leq 100$. |</p>
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Table II
Hénon map with parameters $A = 1.45, \, M = 0.2$; restraining region: $B = \{(x, y): -2 \leq x, y \leq 2\} \setminus D$, where $D$ is a disk of radius 0.005 around the point $(1.172461, -0.070738)$, and the starting interval is the line segment from $(-2,1)$ to $(2,1)$. $N$: the number of subdivisions. $G(N)$: the number of PIM triples in the refinement consisting of three consecutive points of the $N^{-1}$-refinement. $m(N)$: the number of PIM triples consisting of 3 consecutive points including the maximal escape time.

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3.5. Troubleshooting

Recall that we do not guarantee that the described PIM triple procedure will work in all dynamical systems. But it did work in almost all the examples we tried. In cases where the chaotic set has two or more positive Lyapunov exponents, the method is likely to fail. One can ask: What can one do when the PIM triple procedure runs for a while, and thereafter gets stuck?

Some suggestions:

1. Make sure that the restraining region contains no attractors.
2. Make the restraining region bigger.
3. Try another initial interval $[a, b]$; it must be chosen so that it crosses the stable set.
4. Select a larger number $N$ of subdivisions.
5. When a failure occurs, restart from a previously stored PIM triple, and use a slightly different $N$.

4. Analysis of the PIM triple procedure: the Smale horseshoe map

To obtain a better understanding of why the PIM triple procedure works, we analyze horseshoe maps. Let the box $B$ be the unit square, i.e., $B = [0,1] \times [0,1]$. Let $F$ be a differentiable invertible map from the plane into itself, so that $F$ restricted to $B \cap F^{-1}(B)$ is piecewise linear and this set's image $F(B) \cap B$ consists of two components. Let $H_1$ and $H_2$ be the horizontal strips with $F(H_1) = V_1$ and $F(H_2) = V_2$ as indicated in fig. 10(a). We assume, that on $H_1$ and $H_2$ the map $F$ contracts horizontal segments by a factor $\theta$, where
$0 < \theta < 1/2$, $F$ expands on $H_1$ vertically by a factor of $\alpha$, where $\alpha > 2$, and $F$ expands on $H_2$ vertically by a factor of $\tau$, where $\tau > 2$. We write $\lambda = \max\{\alpha, \tau\}$ for the maximum value of the expansion factor. The points in the union of the horizontal strips $H_1 \cup H_2$ that will be mapped back into $H_1 \cup H_2$ by applying $F$ are contained in the set $F^{-2}(B \cap F(B) \cap F^2(B))$ which consists of 4 thinner horizontal strips, see fig. 10(b). By induction, the set $F^{-N}(B \cap F(B) \cap \cdots \cap F^N(B))$ is the set of points in $H_1 \cup H_2$, that will be mapped in $H_1 \cup H_2$ by applying $F^N$, consists of $2^N$ horizontal strips, for each positive integer $N$.

Obviously, $T(x) \geq 1$ for $x$ in $B$ and $T(x) \geq 2$ for $x$ in the 2 horizontal strips $F^{-1}(B \cap F(B))$; for $x$ in the 4 horizontal strips $F^{-2}(B \cap F(B) \cap F^2(B))$ we have $T(x) \geq 3$. Continuing, $T(x) \geq N + 1$ for $x$ in the $2^N$ horizontal strips $F^{-N}(B \cap F(B) \cap \cdots \cap F^N(B))$. In each of these $2^N$ horizontal strips there are two smaller strips of points that stay inside $B$ for at least $N + 1$ iterates, i.e., $T(x) \geq N + 2$ for $x$ in these smaller strips. These two smaller strips are smaller by a factor of at most $\lambda^{-1}$ than the strip consisting of the points whose escape time is at least $N$ and which includes these smaller strips.

The ideas behind the proof of the PIM triple procedure (the "static" restraint problem) are:

1. the interval associated with any PIM triple intersects the stable set $S(B)$;
2. every suitably defined grid of points which is a refinement of a PIM triple includes a PIM triple; the new PIM triple is "proper" so the interval associated with this new PIM triple is by definition smaller than the interval associated with the previous PIM triple;
3. repeatedly applying (1) and (2) with suitable grids of points gives convergence to a point of the chaotic saddle.

Let $J$ be a vertical segment in $V_1$ such that $J = \{z\} \times [0,1]$ for some $z$ with $0 < z < 1$. Obviously, $J$ intersects the stable set.

Let $(a, c, b)$ be a PIM triple in $J$, and let $G$ be a given grid of points on $J$, being an $\varepsilon$-refinement of the triple $(a, c, b)$ and $G \subset [a, b]$. We want to determine how small $\varepsilon$ must be, in order that the three properties mentioned above will hold for every PIM triple in $J$. The assumption that $G$ includes these 3 points $a$, $c$, and $b$ does not slow down the computation of escape time, since their escape times are already computed. We would like to find an $\varepsilon$ sufficiently small that $G$ includes a PIM triple; recall that this means that there are three points $a_1$, $b_1$, and $c_1$ in $G$ such that $T(c_1) > T(a_1)$, $T(c_1) > T(b_1)$, and $|b_1 - a_1| < |b - a|$. If there is any grid point $d$ other than $a$ or $b$ whose escape time is less than $T(c)$ then we are done, since we may let $c_1 = c$, and we may let either $a_1 = d$ and $b_1 = b$ or $a_1 = a$ and $b_1 = d$. If there is a point $d$ whose escape time is bigger than $T(c)$, then we are done, since $c$ can be made an end point of the new PIM triple (either $(a, d, c)$ or $(c, d, b)$). The idea behind the proof of the property that an $\varepsilon$-refinement of a PIM triple includes a PIM triple is to show that at least one of the grid points of $G$ must have escape time different from $T(c)$, where $\varepsilon$ is suitably chosen and which depends only on $\lambda$, the distance between the two horizontal strips $H_1$ and $H_2$ in $B$, and the distance between the horizontal strips and the bottom line and top line of the box $B$.

We may assume that the refinement does not hit the stable set $S(B)$ since otherwise we would be done. Thus we may assume that the escape time of the points in the refinement of a PIM triple is finite. We observe that the escape time function has the following property, whose proof is left to the reader.

No-interior-max property. Let $L \subset J$ be an interval whose interior contains no point of the stable set $S(B)$. Then there is no triple $(a, c, b)$ in $L$ for which $T(c)$ is strictly larger than both $T(a)$ and $T(b)$.

We know that the interval $J$ intersects the stable set $S(B)$ (which is closed). Will the same property hold for each subinterval $L$ of $J$? Clearly, the answer on this question is no, since one can choose the interval $L$ lying entirely in the gap
between the two horizontal strips. Our first result
“PIM existence proposition” characterizes the in-
tervals intersecting the stable set \( S(B) \). The proof
of this proposition is given in the appendix. The
underlying idea is based on fact 3 in the intro-
duction.

**Proposition 1 (PIM existence).** Let \( L = [a, b] \) be
an interval in \( J \). Then the following are equiva-
lent:

(i) there exists an \( \epsilon > 0 \) such that every \( \epsilon \)-refine-
ment of \( a \) and \( b \) includes a PIM triple;

(ii) \( L \) contains a point of \( S(B) \) in its interior.

In proposition 1 the interval \( L \) can be chosen so
that it intersects \( S(B) \) only at points extremely
close to the end points of \( L \) and so \( \epsilon \) would have
to be extremely small, so \( \epsilon \) depends on the choice
of \( L \). For example, if \( L = [a, b] \) with \( a \) in the gap
between \( H_1 \) and \( H_2 \) and \( b \) in \( H_2 \), and \( s \) the point
on \( L \) which is on the boundary of \( H_2 \) (i.e., the
interval \( [a, s] \) would be in the gap between \( H_1 \)
and \( H_2 \)), then it is necessary to have \( \epsilon < |b-s| \).
However, proposition 2 (which is stated below)
shows that a single (that is, uniform) \( \epsilon \) can be
used, for all succeeding PIM triples once we have
found our first PIM triple. In principle it can be
difficult to find the first PIM triple if the initial
interval \( L \) is chosen badly. In practice such bad
choices of \( L \) are very unlikely; the choice of \( L 
\)
with end points on opposite sides of the restraining
region seems to work fine. The main objective
of the paper is to describe how to proceed after a
reasonable \( L \) has been chosen since the actual
choice of \( L \) will be heavily system dependent.

From now on, we will fix \( \epsilon \) and it is defined as

\[
\epsilon = \delta (1 - 2\delta),
\]

where \( \delta \) is the minimum of

1. \( \lambda^{-1} \);
2. the distance between the horizontal strips \( H_1 \)
   and \( H_2 \);
3. the distance between the strip \( H_1 \) and the bot-
tom line of \( B \); and,

4. the distance between the strip \( H_2 \) and the top
   line of \( B \).

**Proposition 2 (PIM refinement).** Every \( \epsilon \)-refine-
ment of a PIM triple in the interval \( J \) includes a
PIM triple.

From this result's constructive proof (given in
the appendix), in each \( \epsilon \)-refinement of a given
PIM triple one may expect to find several PIM
triples, each consisting of three consecutive points.
Recall that in the example of the Hénon map
considered in section 3, we start with the interval
having the end points on the opposite sides of the
restraining region, and the \( \epsilon \)-refinement includes
10 PIM triples which consist of 3 consecutive
points, where \( \epsilon = 1/30 \).

Under reasonable choices of a sequence of PIM
triples \((a_n, c_n, b_n)\) on \( J \) with nested intervals
\([a_{n+1}, b_{n+1}] \subset [a_n, b_n]\), we will have \( |a_n - b_n| \to 0 \)
as \( n \to \infty \).

**Proposition 3 (PIM convergence).** Every nested
sequence of PIM triple intervals in \( J \), whose
lengths go to zero, converges to a point on \( S(B) \).

The proof follows from that the nested sequence
of PIM triple intervals converges to a single point;
this point belongs to \( S(B) \) because each interval
in the sequence has a nonempty intersection with
the stable set, and the intersection \( S(B) \cap J \) is
compact.

From the foregoing it follows that the PIM
triple procedure of the “Static” Restraint Problem
on \( J \) will yield a limit point on the stable set
\( S(B) \). The trajectory of this point will stay in the
box \( B \) forever.

We would like to emphasize, that we will obtain
such a limit point for the horseshoe map whenever
the starting interval \( J \) crosses the stable set
\( S(B) \) – and the horseshoe map seems to be fairly
typical. In particular, \( J \) may have any slope dif-
ferent from zero. This property is important for
the “Dynamic” restraint problem.
From now on, let $I_n$ be the $n$th tiny interval in the numerical trajectory as discussed in section 2 with $I_{n+1} \subset F(I_n)$. We write $d_n$ for the distance between $I_n$ and the unstable set $U(B)$. We can show $d_{n+1} \leq (\theta + 10^{-8})d_n < 0.75d_n$. It follows that all the points in such a tiny interval $I_n$ of length less than $10^{-8}$ have (after some iterates) a distance less than $2 \times 10^{-8}$ to the chaotic saddle $C = S(B) \cap S(U))$. The segments $F^{-n}(I_n)$ converge to a point of $S(B)$ as in the "Static" Restraint Problem. Therefore, the "Dynamic" restraint problem is solved once the "Static" restraint problem has been settled.

5. The accessible PIM triple procedure

We will refer to the complement $R \setminus S(R)$ of the stable set $S(R)$ in the restraint region $R$ as the transient set. Following the notion of accessibility in Grebogi et al. [4] and Alligood and Yorke [1], we will say that a point $p$ in $S(R)$ is accessible from the transient set $R \setminus S(R)$ if there is a continuous curve $K$ ending at $p$ so that all of $K$ except $p$ is in the transient set $R \setminus S(R)$. Our primary interest is in finding accessible trajectories, that is, trajectories for which every point is accessible. As in previous sections, we change this question to the "dual" problem to make the theory simpler. Then we return at the end of the section to the trajectory problem. We now address the following question:

Accessible Restraint Problem. Given a segment $J$ that crosses the stable set $S(R)$, describe a procedure for finding a point in $J \cap S(R)$ which is accessible (from $R \setminus S(R)$).

We will in fact find a point $p$ in $J \cap S(R)$ that is accessible in the most obvious way: there is a point $r \in J$, $r < p$, so that there are no points of $S(R)$ in $(r, p)$. Then the segment from $r$ to $p$ is a curve $K$. While $J$ can intersect $S(R)$ in an uncountable set (a Cantor set), the accessible points are typically only a countable subset of this intersection. The accessible point $p$ we find will be accessible using the curve $K = [r, p]$ that is a segment in $J$ for some $r \in J$, so we say $p$ will be "accessed from the left", that is from the side containing $r$.

Our objective is to describe a procedure (called the accessible PIM triple procedure) that selects in a unique way a nested sequence of PIM triple intervals on $J$ leading to an accessible point in $S(R)$ on $J$. This point is accessible from the left. In practice we observe that the trajectories through accessible points converge rapidly to (unstable) fixed points or periodic orbits. We could alternatively have chosen to approach from the right and we would generally expect to find a different point.

Accessible PIM triple procedure. The strategy for finding successive PIM triple intervals $[a_n, b_n]$ is based on a procedure in which we choose $b_{n+1}$ in $[a_n, b_n]$ "as far left as possible" while $a_{n+1}$ is only slightly to the right of $a_n$. "As far left as possible" means that given a grid $G_n$ which refines $[a_n, b_n]$, we choose $b_{n+1}$ to be the grid point which is as far left as possible and yet is the right end of a PIM triple of grid points. Such a point will be called the beta point of the grid $G_n$. If $b_{n+1}$ is the beta point in the grid $G_n$ and $c_{n+1}$ is the adjacent point in $G_n$ to the left of $b_{n+1}$, there will be a grid point $a_{n+1}$ in $G_n$ such that $(a_{n+1}, c_{n+1}, b_{n+1})$ is a PIM triple. The systematic choice of $a_{n+1}$ for which $(a_{n+1}, c_{n+1}, b_{n+1})$ is not only a PIM triple, but that gives a procedure converging to an accessible point, requires a delicate construction because we must be sure that the intervals $[a_n, a_{n+1}]$ contain no points of $S(R)$ at least for $n$ sufficiently large.

From now on $\epsilon = \delta(1 - 2\delta) > 0$ as in the PIM refinement proposition in section 4. We will assume that the distance between two consecutive points in an $\epsilon$-refinement of two points $a_n$ and $b_n$ in $J$ is at least $\epsilon/2$ times the length $|b_n - a_n|$, and (of course) at most $\epsilon$ times the length $|b_n - a_n|$.
We will refer to such a refinement as a regular \( \varepsilon \)-refinement. Recall that a nested sequence of PIM triple intervals, obtained from regular \( \varepsilon \)-refinements, converges to a point on \( S(R) \) (PIM convergence proposition).

The procedure (that seems to work fine for many dynamical systems) consists of three steps. Given a PIM triple \((a_n, c_n, b_n)\), we choose a regular \( \varepsilon/3 \) refinement \( G_n \) on \( J \), where \( G_n = \{ x_i: 0 \leq i \leq N(\varepsilon) \} \) with \( a_n = x_0 < x_1 < \cdots < x_{N(\varepsilon)} = b_n \) such that \( G_n \) includes \( c_n \). We will choose \( c_{n+1} \) and \( b_{n+1} \) as above with \( b_{n+1} \) the beta point of \( G_n \). Let \( M_n \) be the minimum value of \( \{ T(x_i): x_i \in G_n, x_i < c_{n+1} \} \).

Case 1. If \( M_n < T(x_0) \), then choose \( a_{n+1} \) to be any point \( x_i \) in \( G_n \cap [a_n, c_n] \) for which \( T(x_i) = M_n \). (For many systems \( M_n = T(x_0) \) will hold after the first several values of \( n \).)

Case 2. If \( M_n = T(x_0) \) and \( |c_{n+1} - a_n| \leq |b_n - a_n|/3 \), then choose \( a_{n+1} = a_n \).

Case 3. If \( M_n = T(x_0) \) and \( |c_{n+1} - a_n| > |b_n - a_n|/3 \), then choose \( a_{n+1} \) to be the rightmost grid point \( x_i \) in \( G_n \) such that \( |c_{n+1} - x_i| \geq |b_n - a_n|/3 \).

Notice that since \( N > \varepsilon/3 \), we can expect \( |b_{n+1} - a_{n+1}| \) to be about \( |b_n - a_n|/3 \) or less.

The proof that \( \{ a_n \} \) converges to an accessible point for the horseshoe example is based on arguments of previous sections, and is omitted. This conclusion also holds for uniformly hyperbolic systems, and that result will be proved elsewhere.

This "static" accessible problem's solution leads to the following numerical procedure for the "dynamic" problem.

The numerical procedure we use for finding accessible trajectories goes as follows: Choose a starting interval \( L \) as before; and apply the left accessible PIM triple procedure repeatedly until the length of the interval is less than some distance which we again denote as \( 10^{-8} \), call this interval \( I \). Repeatedly iterate the interval \( I \) forward; whenever the length of the interval \( I \) under forward iteration exceeds \( 10^{-8} \), apply the left accessible PIM triple procedure to decrease its length. Just as we previously obtained a saddle straddle trajectory, here we obtain an accessible saddle straddle trajectory.

In practice when writing programs we will assume that \( G \) is the grid of end points arising in the partition of the interval \([a, b]\) into \( N \) subintervals of equal length, for some integer \( N \geq 3/\varepsilon \). We typically choose \( N = 30 \), that is, we divide the interval into 30 subintervals of the same length. Note that the grid of end points of these 30 smaller intervals is a regular \( \varepsilon \)-refinement with uniform spacing.

For the examples of section 3 we find the accessible saddle straddle trajectory always converges to an accessible fixed point. The points whose trajectories converge to a fixed point are the points of its stable manifold. This suggests at least in the cases we have examined, that all the accessible trajectories in the stable set \( S(R) \) are on stable manifolds of accessible fixed points or of accessible periodic points.

Appendix

The purpose of this appendix is to prove the PIM existence proposition, the PIM refinement proposition, and the PIM leftmost refinement proposition, for the horseshoe map.

The analysis of the horseshoe map depends most strongly on the stretching behavior of the map. The arguments are simplified by the fact that if \((x, y)\) is in \( H_1 \cup H_2 \) (defined below), then the vertical coordinate of \( F(x, y) \) depends only on \( y \), and we write this as \( f(y) \). We explicitly write out \( f \), and we take the liberty of defining \( f \) even when \((x, y)\) is not in \( H_1 \cup H_2 \) to simplify the discussion, though the arguments will be independent of how it is defined.

We write \( H_1 = [0, 1] \times [r, s] \) with \( 0 < r < s \), \( H_2 = [0, 1] \times [t, u] \) with \( s < t < u < 1 \). Note that \( s = r + 1/\alpha \), and \( u = t + 1/\tau \). Now we consider \( f \) as a map from the unit interval \([0, 1]\) into the real line having the following properties: \( f(r) = f(u) = 0 \), \( f(s) = f(t) = 1 \), \( f'(x) = \alpha \) for all \( x \leq s \), and \( f'(x) = -\tau \) for all \( x \geq t \); see fig. 11. Hence \( f(x) = \ldots \)
The points in $I$ whose escape time from $I$ equals $k$ will be written as

$$D_k = \{ x \in I: T(x) = k \}, \quad k = 1, 2, 3, \ldots .$$

In particular, $D_1$ consists of the open component $(s, t)$, and the half open half closed components $[0, r)$ and $(u, 1]$. Note that $D_1 = A_1 \setminus A_2$, see fig. 11.

For every integer $k \geq 1$ we have

(a) $A_k = A_{k+1} \cup D_k$;

(b) $I = A_{k+1} \cup \bigcup_{j=1}^{k} D_j$, that is, $I$ is the union of the set of points $A_{k+1}$ whose escape time from $I$ is at least $k+1$, and the set of points $D_j$ whose escape time from $I$ is $j$, where $1 \leq j \leq k$.

For each $k \geq 1$, the components of $A_{k+1}$ and $D_{k+1}$ can be obtained by iterating backward the components of $A_k$ and $D_k$ in fig. 11.

Furthermore, the set $A_\infty$ has measure zero, since for all $k \geq 1$ one has $0 \leq \mu(A_{k+1}) \leq \frac{2}{F}^k$, where $F = \min \{ \alpha, \tau \} > 2$ and $\mu$ denotes the Lebesgue measure, which in this case is the sum of the lengths of the intervals of which it is composed. Consequently, it is reasonable to expect that the escape time of the points in the refinement of two points (or a triple) is finite since the refinement has finitely many points; hence one may assume that the refinement does not intersect the invariant set $A_\infty$; if it should intersect $A_\infty$, the argument is easier and so is ignored.

Let $J$ be a vertical segment in $V_1$ such that $J = \{ z \} \times [0, 1]$ for some $z$ with $0 < z < 1$. Obviously, $J$ intersects the stable set $S(B)$. We would like to emphasize that the invariant set $A_\infty$ satisfies $\{ z \} \times A_\infty = J \cap S(B)$, where $S(B)$ is the stable set of $F$.

To prove the PLM existence proposition, and the PLM (leftmost) refinement proposition for the map $f$ we need somewhat more information about the time escape function on and the relative lengths of the different components of the introduced sets.

From now on we denote the length of an interval $L$ by $|L|$, and we write $D_\infty = \bigcup_{k=1}^{\infty} D_k$ which thus includes only points with $T(x) < \infty$. The
lemma below states that if the value of the escape time function $T$ changes at $x \in D_{\infty}$ then it changes in steps of 1.

**T-jump lemma.** For every $x \in D_{\infty}$ there exists an $\varepsilon > 0$ such that for each $y$ with $|x - y| < \varepsilon$ one has $|T(x) - T(y)| \leq 1$.

**Proof.** Let $x \in D_{\infty}$ be given. We write $D_{\infty}^{\text{int}} = \bigcup_{k=1}^{\infty} \text{Int}(D_k)$, where $\text{Int}(D_k)$ means the interior of $D_k$ for each $k \geq 1$. First, consider the case where $x \in D_{\infty}^{\text{int}}$. Then, by the definitions, $T$ is constant on the component of $D_{\infty}^{\text{int}}$ including $x$. Consequently, there exists an $\varepsilon > 0$ so that $T(y) = T(x)$ for all $y$ with $|x - y| < \varepsilon$.

Now we consider the case where $x \in D_{\infty} \setminus D_{\infty}^{\text{int}}$. Let $M \geq 0$ be the integer for which $f^M(x) \in \{0, 1\}$. From the fact that the points 0 and 1 are mapped outside $I$ it follows that $T(x) = M + 1$. Hence there exists an $\varepsilon > 0$ so that for each $y$ with $|x - y| < \varepsilon$ either $T(y) = M$ or $T(y) = M + 1$.

We conclude that there exists $\varepsilon > 0$ so that for each $x \in D_{\infty}$ and for each $y$ with $|x - y| < \varepsilon$ either $T(x) = T(y)$ or $|T(x) - T(y)| = 1$.

This completes the proof of the T-jump lemma.

**Geometric lemma.** Let $\delta = \min \{ a^{-1}, \tau^{-1}, r, s - t, 1 - u \}$. For each integer $k \geq 1$, the following hold:

(i) every component of $A_k$ contains components of $D_k$ and of $A_{k+1}$;

(ii) for each component $D$ of $D_k \cap A$, one has $|D|/|A| \geq \delta$, and for each component $U$ of $A_{k+1} \cap A$, satisfies $|U|/|A| \geq \delta$, with $A$ an arbitrarily chosen component of $A_k$.

**Proof of the geometric lemma.**

Proof of (i). Let $k \geq 1$ be a given integer. For $k = 1$ we are done, since that $A_1 = I, A_2 = [r, s] \cup [t, u]$, and $D_1 = [0, r) \cup (s, t) \cup (u, 1]$.

Now we assume $k > 1$. Let $A$ be a given component of $A_k$. We have, $f^{k-1}(A) = I$ by the definition of $A_k$. Therefore, $A$ contains three components of $D_k$ and two components of $A_{k+1}$.

Proof of (ii). We prove (ii) by induction. The case $k = 1$ follows immediately from the definition of $\delta$. Let $n \geq 1$ be $\varepsilon 1$ an arbitrarily given integer, and assume the result is proved for $k = n$.

For $k = n + 1$ we have the following: If $A$, $D \subset A$, and $U \subset A$ are components of $A_{n+1}$, $D_{n+1}$, and $A_{n+2} \cap A$, then $f(A)$, $f(D)$, and $f(U)$ are components of $A_n$, $D_n$, and $A_{n+1} \cap f(A)$, respectively. Since $f$ is linear on each component $A$ of $A_{n+1}$, $f'$ is constant and equals $|f(A)|/|A|$ = $|f(U)|/|U| = |f(D)|/|D|$. Hence $|D|/|A| = |f(D)|/|f(A)|$ which is by the induction hypothesis at least $\delta$. The case $|U|/|A|$ is identical, and so the proof is complete.

**Proposition 1 (PIM existence).** Let $L = [a, b]$ be an interval with positive length in $I$. Then the following are equivalent:

(i) there exists $\varepsilon > 0$ such that every $\varepsilon$-refinement of $a$ and $b$ includes a PIM triple;

(ii) $L$ contains a point of $A_{\infty}$ in its interior.

**Proof of "PIM existence proposition".** Let $L$ be as above.

"(i) \Rightarrow (ii)" We assume that there exists $\varepsilon > 0$ such that every $\varepsilon$-refinement of the points $a$ and $b$ includes a PIM triple. If the interior of $L$ does not include a point of $A_{\infty}$, then, by the 'No-Interior-Max Property' in section 4, no $\varepsilon$-refinement of $a$ and $b$ includes a PIM triple. Hence, the interior of $L$ contains a boundary point of $D_k$ for some integer $k \geq 0$. Therefore, the interior of $L$ intersects $A_{\infty}$.

"(ii) \Rightarrow (i)" Now we assume that the interior of $L$ contains a point $q$ of $A_{\infty}$. We write $\Gamma = \max \{ 1/\alpha, 1/\tau \}$. Pick some positive integer $N$ such that $\Gamma^N < \min \{ q - a, b - q \}$. We write $A$ for the component of $A_{N+1}$ containing $q$. Let $\delta > 0$ be as in the geometric lemma. Now we select $\varepsilon = \delta^2 \cdot |A|$. From the geometric lemma we know that $A$ contains three components of $D_{N+1}$ whose length of all three is at least $\delta \cdot |A|$, and $A$ contains at least two components of $A_{N+2}$ whose length is at least $\delta \cdot |A|$. Therefore each $\varepsilon$-refinement of $a$ and $b$ includes a PIM triple in the component $A$.

This completes the proof of the PIM existence proposition.
Recall that the $\varepsilon$ in proposition 1 depends on the given interval $L$, but the $\varepsilon$ where we are interested in will only depend on $\lambda$, on $(t-s)$ (which is the distance between the two horizontal strips), and on the distances between the horizontal strips and the boundary of the region $R$. From now on, we will fix $\varepsilon$ to be $\varepsilon = 8(1-2\delta)$ where $\delta = \min\{a^{-1}, 1, t-s, 1-u\}$, as defined in the geometric lemma.

**Proposition 2 (PIM refinement).** Every $\varepsilon$-refinement of a PIM triple in the interval $I$ includes a PIM triple.

**Proof of “PIM refinement proposition”**. Let $(a, c, b)$ be a PIM triple in $I$. First, we assume that $T(a) \leq T(b) < T(c)$.

Case 1. Assume $k = \min_{a \leq x \leq b} T(x) < T(a)$. Let $D$ be the component of $D_\infty$ containing at least one point of $[a, b]$ for which $T(y) = k$ for all $y$ in $D$. Then $D \subset (a, b) \subset A$ where $A$ is the component of $A_k$ for which $D \subset A$. Since $|b-a| \leq |A|$, applying the geometric lemma gives $|D|/|b-a| \geq |D|/|A| \geq \delta$. Then, for each $\varepsilon$-refinement $G_\varepsilon$ of $(a, c, b)$, with $0 < \varepsilon \leq \delta$, we have $G_\varepsilon \cap D \neq \emptyset$. We obtain: for every $P \in G_\varepsilon \cap D$ either $(P, c, b)$ or $(a, c, P)$ is a PIM triple in $G_\varepsilon$.

Case 2. Assume $\min_{a \leq x \leq b} T(x) \geq T(a)$ and $T(c) \geq T(a) + 2$. Then, by the “$T$-jump” lemma, there exists a component $D$ of $D_{T(a)+1}$ in the interval $[a, c]$. Since $[a, b]$ lies in a component of $A_{T(a)}$ the geometric lemma implies $|D|/|b-a| \geq \delta$. Hence every $\varepsilon$-refinement contains a point $a_*$ of $D$, so $(a_*, c, b)$ is a PIM triple, where $0 < \varepsilon \leq \delta$.

Case 3. Assume $T(c) = T(a) + 1$ and that case 1 does not apply. This implies $T(b) = T(a)$. Set $\alpha = \delta(1-2\delta)$; let $G_\alpha$ be an $\alpha$-refinement of $(a, c, b)$, say $G_\alpha = \{x_i: 0 \leq i \leq N(\alpha)\}$ with $a = x_0 < x_1 < \cdots < x_{N(\alpha)} = b$ and $x_k = c$ for some $1 \leq k \leq N(\alpha) - 1$.

Case 3(a). Assume there is $j$, $1 \leq j \leq N(\alpha) - 1$, such that $T(x_j) \leq T(a)$, then either $(a, c, x_j)$ or $(x_j, c, b)$ is a PIM triple in $G_\alpha$.

Case 3(b). Assume $T(x_j) \geq T(c)$ for all $1 \leq i \leq N(\alpha) - 1$. Let $U$ be the component of $A_{T(a)+1}$ which is included in $[a, b]$. Note that the facts $x_i \in U, 1 \leq i \leq N(\alpha) - 1, |x_0 - x_1| < \delta |b-a|$, and $|x_{N(\alpha)} - x_{N(\alpha)}| < \delta |b-a|$ imply that $|U|/|b-a| > 1 - 2\delta$. From the geometric lemma we get that $U$ contains a component $D$ of $D_{T(a)+2}$, and $|D|/|U| \geq \delta$. Now we have $|D|/|b-a| = (|D|/|U|)(|U|/|b-a|) > \delta(1-2\delta) = \alpha$. Let $x_j \in G_\varepsilon \cap D$; then either $(a, x_j, c)$ or $(c, x_j, b)$ is a PIM triple in $G_\varepsilon$.

We obtain that every $\varepsilon$-refinement of $(a, c, b)$ includes a point from $D$ and thus it includes a PIM triple for each $0 < \varepsilon \leq \delta(1-2\delta)$.

The case $T(b) \leq T(a) < T(c)$ is similar. The conclusion is: For $\varepsilon = \delta(1-2\delta)$ we have: every $\varepsilon$-refinement of a PIM triple in $I$ includes a PIM triple.

This completes the proof of the PIM refinement proposition.

**Remark.** From this constructive proof one may expect several PIM triples in every $\varepsilon$-refinement of each arbitrarily given PIM triple. In practice, there are typically 3 consecutive points that are a PIM triple, (though computer programs should not rely upon such events).

**Remark.** From the foregoing it follows that the PIM triple procedure when applied to the “static” restraining problem on $I$ produces a sequence of PIM triples (with the length of the associated intervals decreasing to 0) which converges to a point $q$ on the invariant set $A_\infty$. Therefore, on the interval $J$ it will converge to the stable set $S(B)$, namely the point $(z, q)$. The trajectory of this point will stay in the box $B$ forever.

References