WHY PERIOD-DOUBLING CASCADES OCCUR: PERIODIC ORBIT CREATION FOLLOWED BY STABILITY SHEDDING

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Period-doubling cascades of attractors are often observed in low-dimensional systems prior to the onset of chaotic behavior. We investigate conditions which guarantee that some kinds of cascades must exist.

1. Introduction

Cascades of period-doubling bifurcations have been seen in the great majority of low-dimension systems that for certain parameter values also exhibit chaotic behavior, that is, sustained irregular aperiodic oscillations. A “cascade” has an infinite sequence of period-doubling bifurcations, a phenomenon observed as a physical parameter is varied by the experimenter. A stable periodic orbit is seen to become unstable as the parameter is increased or decreased and is replaced by a stable periodic orbit of twice its period. This orbit in turn becomes unstable and is replaced by a new stable orbit with a period again higher by a factor of two, and the process continues through an infinity of such period-doubling bifurcations. Of course an experimenter will only be able to see a few of these stages. Renormalization theory [1, 2] has been extremely successful in showing it is possible for the full infinite sequence to exist and in establishing a number of remarkable regularities of this process, yielding universal numbers that are regularly observed in both physical and numerical experiments. These arguments, however, do not provide any intuition as to why these cascades are so prevalent. Cascades consist of stable periodic orbits and so are experimentally observable. Chaotic attractors, on the other hand, contain infinitely many unstable periodic orbits that are essentially not observable. This paper will relate these unstable orbits at one parameter value to cascades that occur in other parameter ranges.

Poincaré was the first to give a complete mathematical description of a period-doubling bifurcation, and Myrberg [4] first described a cascade...
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(An infinite sequence of period-doubled, stable, periodic orbits). Cascades became well known with the applications of May [5]. They were first reported in two-dimensional maps by Derrida et al. [3], and later in the area-preserving case by Derrida (unpublished). Fig. 1 (adapted from Grebogi et al. [3]) is similar to commonly seen cascades. Our emphasis here is on the fact that many distinct cascades are seen or are hinted at in this picture. Numerical studies reveal that cascades occur not in isolation but in great profusion, sprinkled throughout chaotic regimes. (See, for example, section 3 of this paper.)

The purpose of this paper is to give an explanation of why cascades must occur, and why they must occur in great profusion. The explanation is only adequate for chaotic attractors that have one unstable direction at each point (that is, only one positive Lyapunov exponent). Indeed there seems to be no reason to expect to find many cascades in highly chaotic systems which are unstable in more than one direction. Hence we will concentrate here on two-dimensional systems, those with one positive and one negative Lyapunov exponent. Our description is given in terms of a process of orbit creation followed by stability shedding and builds on two well-known phenomena: the coexistence of huge numbers of unstable periodic orbits in a chaotic attractor (for a fixed parameter value), and the general bifurcation phenomenon known as “the exchange of stabilities”. As the parameter is increased, the collection of periodic orbits changes. For example, it might change from a single stable orbit at parameter value $L_0$ to the infinite hoard of unstable orbits one expects in a chaotic attractor at $L_1$. We will argue here that the only way in which these unstable orbits can arise (given the absence of stable orbits at $L_1$) is through the process of cascade formation. Approximately half the orbits in a chaotic attractor may be described as “regular” saddles. We will show that for each of these regular saddles there must be a distinct cascade occurring in some parameter range. Therefore, in changing from a single stable periodic orbit to a chaotic attractor, the system must have had not just one cascade but, in fact, infinitely many cascades.

2. Cascades of attractors

Assume that the dynamical process under consideration can be described by an iterative process,

$$x_{n+1} = f(L, x_n),$$

where $x$ is a point in the plane and $L$ is a scalar parameter. The $k$th iterate of $f$ is denoted $f^k$, and we say $(L, x)$ is a periodic point of period $k$ when $x = f^k(L, x)$. Imagine a small disk lying across a
chaotic attractor in the plane. Characteristically, the map \( f \) will stretch the disk in one direction, contract it in another, and then fold it, so that the \( k \)th image of the ball is a long thin band winding around and through the chaotic attractor. For large \( k \), the length will be proportional to \( c^k \) for some \( c > 1 \) (the stretching factor), and this image can be expected to intersect the original disk at a number of places – the number being proportional to \( c^k \), when \( k \) is large. See fig. 2. Generally, there will be an unstable periodic orbit in each of these intersections, so the number of unstable periodic orbits of period \( k \) inside the original disk can be expected to grow exponentially with \( k \). (The numerical calculation of such periodic orbits for a specific example is discussed in section 3).

The “eigenvalues of a periodic orbit” will refer to the eigenvalues of \( \mathbf{D} f^k(L, x) \), where we always take \( k \) to be the minimum period of the orbit and \( \mathbf{D} f \) means we are taking partial derivatives with respect to the two coordinates of \( x \), not \( L \). (Choosing any of the other \( k - 1 \) points on that orbit in place of \( x \) will give the same eigenvalues.)

We now restrict attention to area-contracting* maps of the plane. In maps with one positive Lyapunov exponent (i.e., stretching in one direction), a region will be stretched into a long thin image; and for that image to fit into the original bounded region, a lot of folding can be expected. The area-contracting hypothesis also restricts the types of periodic orbits that can occur. Orbits are unstable when they have an eigenvalue outside the unit circle in the complex plane. Periodic orbits of area-contracting maps can have only one eigenvalue on or outside the unit circle. Any such eigenvalue must be real, so the unstable orbits are in fact saddle orbits. We say a saddle is “regular” if it has a real eigenvalue larger than \( +1 \). If it has an eigenvalue less than \( -1 \), it is called “a twisted saddle”. Again these are the only possibilities for unstable orbits.

Our methods and conclusions about cascades will also apply to area-preserving maps. The stable orbits in that case are elliptic; that is, both eigenvalues are nonreal and are on the unit circle. Although area-preserving maps have a richer structure of bifurcations (such as period tripling), these additional bifurcations can be ignored when following paths or orbits to track cascades.

For a point \((L_o, x_0)\) of minimum period \( k \), if \( \mathbf{D}[f^k(L_o, x_0) - x_0] \) has non-zero determinant, the orbit will persist for sufficiently small changes in \( L \) (by the implicit function theorem). This non-degeneracy condition is satisfied if \(+1\) is not an eigenvalue of the orbit. As \( L \) is increased or decreased, we can follow this path of orbits as long as \(+1\) is not an eigenvalue. We call this fully extended path a branch of periodic orbits. When \(+1\) is an eigenvalue, a bifurcation will generally occur: branches of orbits meet and terminate at such periodic orbits. These bifurcations are called saddle nodes. When \(-1\) is an eigenvalue, the orbit \((L, x)\), say of period \( k \), can be uniquely continued as described above, (i.e., \((L, x)\) lies on a one-dimensional path of orbits of period \( k \)); nevertheless, \((L, x)\) can be a bifurcation orbit, since \(+1\) is then an eigenvalue of \( \mathbf{D} f^{2k}(L, x) \). An additional branch of periodic orbits of minimum period \( 2k \) can terminate at \((L, x)\), which is then called a period-doubling bifurcation orbit. Unless special symmetries are present at that bifurcation point, the only bifurcations that are likely to be encountered in area-contracting maps are the saddle-node

*When the Jacobian matrix \( \mathbf{D} f \) of partial derivatives satisfies \( \det(\mathbf{D} f) = 1 \) for all \((L, x)\), the map is called “area-preserving” and when that determinant always has absolute value less than 1, the map is “area-contracting”.
Fig. 3. The configurations in (a) denote all the generic bifurcations for area contracting maps. The points \( B^+ \) and \( B^- \) denote saddle-node and period-doubling bifurcation points, respectively. The numbers \( k \) and \( 2k \) denote the period of the branch, and there are no other orbits of period \( k \) or \( 2k \) in the picture, assuming the picture represents a sufficiently small neighborhood of the points \( B^+ \) and \( B^- \). St, RS, and TS denote branches consisting of orbits that are stable, regular saddles, or twisted saddles, respectively. The same configuration occurs at all \( k \) points of the bifurcation orbit. In (b) a schematic version is shown. Each point represents an entire orbit. In the period-doubling pictures in (a), the \( k \)th iterate \( f^k \) maps the top branch of points to the bottom branch and vice versa, so both branches represent the same orbits. Therefore they become a single branch in (b). The numerical analyst following a path of orbits must calculate all the points of an orbit in order to calculate any of them, so the orbit is thought of as an entity. The segments in (b) are referred to as "branches". Twisted saddles are represented as dashed lines since they are ignored when following snakes.

and period-doubling bifurcations shown in fig. 3a. The bifurcations shown in fig. 3 are the "generic" bifurcations.

The map \( f \) is called generic if all its bifurcations are generic. As mentioned in the caption of fig. 3, the points on generic bifurcation orbits of period \( k \) are isolated from each other for fixed \( k \). One consequence of this fact is that there are only finitely many bifurcation orbits of period \( k \) in any bounded region of \((L, x)\)-space for any generic map. However, if we take all periods \( k \) together, there can be infinitely many bifurcation points in a bounded region. We will assume that \( f \) in eq. (1) is generic. While genericity is not necessary for the conclusions of this paper, it does significantly simplify the arguments. Any map \( G \) that is not generic can be approximated by a generic map \( f \). The approximation techniques needed to extend our arguments to the general case (i.e., the case without genericity assumptions) can be found in Yorke and Alligood [6, 7]. That paper contains arguments that are similar to those of the present paper though the objectives there are rather different and are more specific. See also Alligood [8] for other properties of systems with cascades.

We now isolate a path of orbits in \((L, x)\)-space within the possibly vast and interconnected network of orbits that is possible even in the generic case. The path will enable us to follow cascades even when there are numerous saddle nodes, period doublings, and period halving along the path. Fig. 3b is a schematic version of fig. 3a, letting a single point represent all \( k \) or \( 2k \) points on an orbit. All the orbits on a branch shown in fig. 3b are the same type: either stable orbits, regular saddles, or twisted saddles. These branches end at bifurcation orbits, and the bifurcation orbits serve to connect the different types of branches. We may therefore refer to branches as stable branches, twisted saddle branches, etc. We now define particular paths of periodic orbits (called "snakes"). Snakes can contain any periodic orbits that are not twisted saddle orbits. For any point \((L, x)\) on a periodic orbit that is not a twisted saddle, we define the snake through \((L, x)\) to be the collection of stable branches and regular saddle branches that can be reached by a connected path of these branches (and their endpoints) from \((L, x)\). A snake passes through a stable orbit, regular saddle, or bifurcation orbit as a one-dimensional path of orbits. This can be seen by noticing that there are two branches of non-twisted orbits emanating from each bifurcation orbit (cf. fig. 3). A snake can be a closed loop of orbits, but if it is not, then the snake will never go through any orbit twice. The following observation from fig. 3 about period doubling will be useful:

A) When the snake connects a branch of period \( k \) orbits to a branch of period \( 2k \), the period \( k \) branch is always stable.

In each of the period \( k \) bifurcations in fig. 3, precisely one of the period-\( k \) branches is stable. (Where there is a period-\( 2k \) branch, it can be stable or saddle.) The exchange of stability principle is a vaguely formulated rule of thumb de-
scribing what can be expected near many bifurcation points. It applies to a variety of situations including Hopf and symmetry-breaking bifurcations. For our purposes, it can be stated precisely as follows:

B) When the non-twisted branches lie on opposite sides of the bifurcation point, one on the left and one on the right, both branches are stable, while if they lie on the same side, one branch is stable and the other is a branch of regular saddles. We discuss the exchange of stability in terms of degree theory in section 4.

Since snakes are composed of branches of non-twisted periodic orbits, the direction of travel along a snake, i.e., whether \( L \) is increasing or decreasing, will change precisely when moving from a stable branch to a regular saddle branch or vice versa. Suppose \( (L_c, x_c) \) is a point on a regular saddle, and suppose we follow the snake (possibly through many reversals in direction). If the snake ever returns to \( L = L_o \), then upon the first return, our direction of travel will be reversed: \( L \) will be decreasing if we were initially increasing it at \( (L_c, x_c) \), and \( L \) will be increasing if we were initially decreasing it, so there have been an odd number of reversals. An odd number of reversals has the net implication that we have travelled from our initial branch of saddles to a stable branch. See fig. 4. Hence the new orbit we have found at \( L_c \) must be stable.

If we assume that the dynamics at \( L_c \) are chaotic, so that there are lots of saddles, and we assume there are no stable orbits, we conclude:

C) Any snake that phases though a chaotic value \( L_c \) will never lead back to \( L_c \).

Let us now make some assumptions that are in accord with what appears to be true for many typical maps that exhibit chaos: (1) there is a chaotic attractor at \( L_c \); (2) at \( L_c \) there are infinitely many periodic orbits (counting orbits of all periods), all of which are saddles, and these include infinitely many regular saddles; (3) at some other value, \( L_0 \), there are finitely many periodic orbits; and (4) for \( L \) between \( L_0 \) and \( L_c \), the periodic orbits all lie in a bounded region of the plane.

Choose any regular saddle at \( L_c \) and follow its snake toward \( L_0 \). From (C) the snake cannot return to \( L_c \). It can lead to \( L_0 \), thus connecting our initial orbit to an orbit at \( L_0 \). No other orbit at \( L_c \) can connect to the same orbit at \( L_0 \), since snakes are disjoint paths. We have assumed there are only a finite number of orbits at \( L_0 \), so there are infinitely many regular saddles that do not lead to \( L_0 \). Their snakes are bound to wander between \( L_0 \) and \( L_c \). Such a snake must consist of infinitely many branches, since otherwise there would be a final branch and that would end in a bifurcation orbit. That is impossible since every bifurcation orbit has two branches of non-twisted orbits emanating from it. In following the snake we pass through infinitely many bifurcation orbits. However, since the map \( f \) is generic we must eventually pass the last bifurcation orbit having period \( N \) or less. In other words: the periods of orbits along the snake go to infinity (though not necessarily monotonically, since at each bifurcation the period can double, remain the same, or be cut in half). Therefore:

D) En route the snake must pass through period-doubling bifurcation orbits of period \( k, 2k, 4k, 8k, \ldots \), and by (A) the snake passes through stable branches of all these periods.

Fig. 4. A snake that contains a regular saddle at \( L_c \) and then returns to \( L_c \) must arrive back at a stable orbit. Thick curves denote stable branches, thin curves denote regular saddles, and dashed lines, twisted saddles.
We may reasonably take (D) to be a characterization of a cascade: a connected path of orbits containing stable branches of period \(k, 2k, 4k, \ldots\), where \(k\) is the period of the orbit with which the snake begins. Of course there can be more than one stable branch of the same period, and there could even, for example, be stable branches of periods \(k/2\) and \(k/4\), assuming \(k\) is divisible by 4.

There are extra regular saddles at \(L_c\); that is, saddles whose snakes do not connect to orbits at \(L_0\). But every regular saddle at \(L_c\) has its own distinct snake. Therefore:

E) For every extra regular saddle at \(L_c\), there is a distinct cascade. Each cascade is connected to at most one regular saddle at \(L_c\).

We have thus argued that the only way extra orbits can be added is by having a cascade*. There is no other way orbits can be created and shed their initial stability—provided the mapping is area contracting or area preserving. If, on the other hand, orbits could become unstable by having a pair of eigenvalues cross the unit circle, cascades need not occur. The assumption that \(x\) is two-dimensional is used only to guarantee that an orbit can have at most one eigenvalue outside the unit circle. This condition is trivially satisfied for the cases we considered here and also for one-dimensional maps. Higher-dimensional cases where cascades are seen are systems where this property holds on the cascade.

3. A numerical investigation

Fig. 5 shows schematically the results of an effort to track the fate of several low-period orbits of a two-parameter map introduced by Curry and Yorke [10]. The map is specified by the composition of two invertible maps in the plane. Letting \(p\) and \(\theta\) be polar coordinates and \(x\) and \(y\) Cartesian coordinates, define

\[
\begin{align*}
\phi_1(p, \theta) &= (e \ln (1 + p), \theta + \theta_0), \\
\phi_2(x, y) &= (x, y + x^2), \\
\phi &= \phi_2 \circ \phi_1.
\end{align*}
\]

The parameter \(\theta_0\) was held constant at \(\theta_0 = 2\) while \(\epsilon\) was varied within the range shown in fig. 5. Orbits were tracked using the Newton method and their stability was ascertained by computing the Lyapunov exponent. Among the interesting—but probably quite common—features observed here are the following (see fig. 5) ones:

1) The existence of period-doubling cascades for both increasing and decreasing \(\epsilon\). In fact we conjecture that for any map of the plane with nonzero Jacobian, if there are cascades then there will be some cascades going in the reverse direction. See, for example, Franceschini [11] for the Henon map.

2) The simultaneous existence of several stable periodic orbits in some parameter ranges.

3) The persistence of many unstable periodic orbits in regions where the map has a chaotic attractor.
4) A lack of relationship between the parameter value at which a period-k orbit terminates and that at which the period-2k orbit which arises from it in a cascade terminates. (E.g., period 14 arises at A via period doubling but does not terminate near B where period 7 terminates. Instead it terminates far from B via a saddle-node bifurcation.)

The presence of these periodic orbits can easily be numerically observed for sufficiently low periods. If the map returns after k iterates to a small neighborhood of an initial point which has been chosen on or near the chaotic attractor, a technique such as Newton's method can be used to search for a nearby orbit of period k. Using this procedure we were able to find 16 distinct unstable orbits with periods less than 50 passing through a single typical disk of diameter 0.01 on the attractor for system (2). Verification of a higher-period orbit with a very large Lyapunov exponent would require more refined numerical methods.

4. Some remarks on exchange of stability and degree

The exchange of stability result in section 2 depends on facts from degree theory. Degree is defined mostly for regular values: Given a compact region $\Omega$ in n-dimensional Euclidian space $\mathbb{R}^n$ and a map g defined on $\Omega$ (into $\mathbb{R}^n$), z is a regular value of g if $Dg(z_0)$ is non-singular at each point $z_0$ in $g^{-1}(z)$. (Here $Dg$ is the Jacobian matrix of partials of g.) The degree of g with respect to $\Omega$ about the point $z$ is the number of pre-images of z in $\Omega$ for which $Dg(z_0)$ has positive determinant minus the number for which $Dg(z_0)$ has negative determinant. For ease of exposition we will restrict ourselves to consideration of the degree about $z = 0$.

The degree can be determined simply with knowledge of the function on the boundary of $\Omega$, and hence can be computed even when 0 is not a regular value. For maps of the plane, which is our focus, it is simply the winding number $W(g, C)$ of g about a simple closed contour C (the boundary of $\Omega$). The theory is quite reminiscent of the derivation of the Cauchy integral formula. For $g = (g_1, g_2)$,

$$W(g, C) = \frac{1}{2\pi} \oint_C \frac{g_1 dg_2 - g_2 dg_1}{|g|^2}$$

$$= \frac{1}{2\pi} \oint_C d\text{arg},$$

where

$$\text{arg} = \arctan \frac{g_2}{g_1}$$

This integral measures the number of times the vector $g(z_0)$ makes a complete $(2\pi)$ rotation as $z_0$ moves continuously around the contour. For example, consider the map of the complex plane given by $g(z) = z^2$, and let C be the unit circle. Then $W(g, C) = +2$. (Notice that 0 is not a regular value of g.) Since we begin and end at the same point on C, the winding number is always an integer. The orientation of the rotation is also taken into account, hence the degree can be negative.

Notice that the integral is defined only when g is non-zero everywhere on the contour C. In addition, its value varies continuously with continuous changes in g and dg. However, since $W(g, C)$ can take on only integer values, this means that the degree is invariant under perturbations of g which introduces no zeroes on C. (Analogously, it is invariant under deformations of C which do not cross zeroes of g.) Summarizing, we list some properties of the degree:

D1) If a simple closed curve $C_1$ can be continuously deformed to another simple closed curve $C_2$ without crossing any zeroes of g, then $W(g, C_1) = W(g, C_2)$.

D2) As long as no zeroes of g intersect C then $W(g, C)$ remains constant through changes in a parameter.

D3) If there are no zeroes of g inside C, then $W(g, C) = 0$. (In this case, darg is an exact differential, and the contour integral is 0.)
We say a zero $z_0$ of $g$ is isolated if there exists a contour $\Gamma$ such that $z_0$ is inside $\Gamma$ and it is only zero of $g$ inside $\Gamma$. We write $W(g, z_0)$ for $W(g, \Gamma)$ in this case and call $W(g, z_0)$ the local degree of $g$ at $z_0$. Of course, if $\det Dg(z_0)$ is non-zero, this number will be $\pm 1$. From (D1) we obtain:

D4) If each zero of $g$ in $C$ is isolated, then $W(g, C)$ is the sum of the local degrees of the zeroes.

We use degree theory to study fixed points of an area-contracting map $f$ by setting $g(z) = z - f(z)$. Then $z_0$ is a zero of $g$ if and only if $z_0$ is a fixed point of $f$. For a fixed point $z_0$ of $f$, we define the fixed point index $I(f, z_0)$ to be the local degree of $g$ at $z_0$. If $z = 0$ is a regular value of $g$, then we can evaluate $I(f, z_0)$ by simply checking the sign of $\det Dg(z_0)$. It is an easy exercise to show that this sign is determined by the number of eigenvalues of $Df(z_0)$ which have real part greater than $+1$. In this way we see that $I(f, z_0) = (-1)^{\sigma}$, where $\sigma$ is the number of such eigenvalues. Of course, if $0$ is not a regular value of $g$ (equivalently, if $+1$ is an eigenvalue of $Df(z_0)$, for some fixed point $z_0$), then more work would be required.

Now we examine the principle of exchange of stability using degree theoretic properties (D2) and (D4). Each of our possible generic non-bifurcation fixed points (attractor $A$, regular saddle $S$, and twisted saddle $T$) has the property that $+1$ is not an eigenvalue. (Notice that stable orbits are attractors in area-contracting maps.) Hence, we easily calculate:

$I(f, A) = +1,$
$I(f, S) = -1,$
$I(f, T) = +1.$

It is important to note that although the twisted saddle is unstable, it has index $+1$, like the attractor.

In examining the period-doubling bifurcation going from period 1 (fixed points) to period 2, first notice (fig. 3) that the two period-1 branches are $A$’s and $T$’s—both having index $+1$. For each parameter value, these points are isolated from other fixed points (and the degree is seen to remain constant along this path). A change in degree only occurs when we look at the map $f^2$. Since a periodic point of period 2 is a fixed point of $f^2$, the theory can be applied to these points also, using $g(x) = x - f^2(x)$. More generally, a periodic point of $f$ of period $n$ is a fixed point of $f^n$; and if $\lambda$ is an eigenvalue of $Df(z_0)$, then $\lambda^n$ is an eigenvalue of $Df^n(z_0)$, for a fixed point $z_0$ of $f^n$.

In analyzing the index of $f^2$ for fixed points of $f$, notice that each twisted saddle of $f$ (having a real eigenvalue less than $-1$) becomes a regular saddle of $f^2$ (that eigenvalue squared is greater than $+1$). Each attractor of $f$ remains an attractor for $f^2$. Thus we see that the index of $f^2$ along the path of fixed points going through a period-doubling bifurcation changes from $-1$ to $+1$ (or vice versa) at the bifurcation orbit. In order to preserve the degree as the parameter changes, we must have another branch of orbits of period 2 bifurcating here. Since there are no other fixed points, these orbits must have minimum period 2. The type of these orbits can be determined by adding indices (property D4). If the bifurcating branch is on the same side of the bifurcation orbit as the attractors, then it must have index $-1$ and be a branch of period 2 regular saddles. (Counting, we have $+1$ for the attractor and $-1$ for each of the two period-2 saddles—totaling $-1$, the index of the twisted saddles on the other side of the bifurcation orbit.) Analogously, if the bifurcating branch is on the same side as the branch of twisted saddles, then in order to preserve degree, it must be a branch of attractors. When considering higher-period orbits (i.e., period $n$ to period $2n$), the analysis is the same if we look at the local degree at one point on the orbit. This proves the exchange of stability principle.

References