BASIN BOUNDARY METAMORPHOSES: CHANGES IN ACCESSIBLE BOUNDARY ORBITS*

Celso GREBOGI, a Edward OTT a,b and James A. YORKE c
University of Maryland, College Park, MD 20742, USA

Received 5 November 1985
Revised manuscript received 28 April 1986

Basin boundaries sometimes undergo sudden metamorphoses. These metamorphoses can lead to the conversion of a smooth basin boundary to one which is fractal, or else can cause a fractal basin boundary to suddenly jump in size and change its character (although remaining fractal). For an invertible map in the plane, there may be an infinite number of saddle periodic orbits in a basin boundary that is fractal. Nonetheless, we have found that typically only one of them can be reached or "accessed" directly from a given basin. The other periodic orbits are buried beneath infinitely many layers of the fractal structure of the boundary. The boundary metamorphoses which we investigate are characterized by a sudden replacement of the basin boundary's accessible orbit.

1. Introduction

1.1. Preliminary discussion of basin boundary metamorphoses

Basin boundaries* for dynamical systems with multiple attractors can be either smooth or fractal; they can appear to be very convoluted; or they can be simple in appearance. As an example illustrating this, fig. 1 shows some computer-generated pictures of basins of attraction for the damped driven pendulum, \( \dot{\theta} + \nu \dot{\theta} + \omega^2 \sin \theta = f \cos t \). Given the variety of basin boundary structure observed in fig. 1, it is natural to ask how basins of attraction change as a system parameter varies continuously. In particular, are there qualitative changes of basin boundaries as the parameter passes through certain critical values? This paper is devoted to a study of such changes, which we call basin boundary metamorphoses. In particular, basin boundary metamorphoses can occur as a result of a homoclinic tangency of the stable and unstable manifolds of a saddle orbit on the basin boundary.

Perhaps the simplest nontrivial system exhibiting the phenomena of interest here is the Hénon map, and so our numerical investigations are devoted to this case. More generally we believe that the type of metamorphoses we investigate are the typical large scale metamorphoses of maps in two dimensions and in many ordinary differential equation systems (e.g., the forced damped pendulum). The Hénon map is

\[
\begin{align*}
  x_{n+1} &= A - x_n^2 - J y_n, \\
  y_{n+1} &= x_n,
\end{align*}
\]

where the parameter \( J \) is the determinant of the Jacobian matrix of the map. (Hence \( J \) is the net contraction ratio for any finite area in the \( x-y \) plane under the action of the map.) For eqs. (1), as \( A \) increases past the value \( A_1 = -(J+1)^2/4 \), a saddle-node bifurcation occurs in which one at-
Fig. 1. Basins of attraction for the damped driven pendulum, $\ddot{\theta} + \nu \dot{\theta} + \omega^2 \sin \theta = f \cos t$. To generate each one of these pictures, we choose over 1 000 000 initial conditions in a grid. The differential equation is then integrated for each initial condition until the orbit is close to one of the possible attractors. For a given picture, if an orbit goes to the attractor indicated below, a black dot is plotted corresponding to that initial condition. Thus, the black region in each picture is essentially the basin of attraction for that attractor.

(a) $\omega = 1$, $\nu = 0.1$, $f = 1.2$; attractor in the black basin: $\theta = -2.2055$, $\dot{\theta} = 0.3729$. (b) $\omega = 1$, $\nu = 0.1$, $f = 2.0$; attractor in the black basin: $\theta = -0.8058$, $\dot{\theta} = 0.9375$. (c) $\omega = 1$, $\nu = 0.2$, $f = 1.8$; attractor in the black basin: $\theta = 2.9320$, $\dot{\theta} = 0.5200$. (d) $\omega = 1$, $\nu = 0.3$, $f = 1.85$; attractor in the black basin: $\theta = 0.4186$, $\dot{\theta} = -0.5744$.

tracting fixed point and one saddle fixed point are created (cf. table I). For $A$ just slightly larger than $A_1$ there are two attractors, one at $(|x|, |y|) = \infty$, and the other the attracting fixed point (for $A < A_1$, infinity is the only attractor). As $A$ increases past $A_1$, the fixed point attractor moves away from the saddle fixed point. The saddle fixed point lies on the boundary of the basin of attraction of the fixed point attractor. In fact, the stable manifold of the saddle is the basin boundary (cf. fig. 2a). As
Table I
Notation for special values of the Hénon map parameter $A$

<table>
<thead>
<tr>
<th>Notation</th>
<th>Event signified</th>
<th>Where notation first appears</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_o$</td>
<td>saddle-node bifurcation of a period-$n$ orbit.</td>
<td>section 1.1</td>
</tr>
<tr>
<td>$A_{sf}$</td>
<td>smooth-fractal metamorphosis</td>
<td>section 1.1</td>
</tr>
<tr>
<td>$A_H$</td>
<td>fractal-fractal metamorphosis</td>
<td>section 1.1</td>
</tr>
<tr>
<td>$A_s$</td>
<td>chaotic attractor has a crisis by colliding with an accessible period-$n$ saddle on the basin boundary</td>
<td>section 1.3</td>
</tr>
<tr>
<td>$A_s^*$</td>
<td>tangency of the stable and unstable manifolds of an accessible period-$n$ saddle on the basin boundary</td>
<td>section 2.1</td>
</tr>
<tr>
<td>$A_{s,n-1}$</td>
<td>tangency of the unstable manifold of a period-$n$ saddle with the stable manifold of a period-$n+1$ saddle</td>
<td>section 3.3</td>
</tr>
</tbody>
</table>

As $A$ increases further, the fixed point attractor undergoes various bifurcations, most prominently a period-doubling cascade. We shall be interested in following the evolution of the basin boundary that is created at $A = A_1$ as the parameter $A$ increases.

Figs. 2 show the basin of attraction for the point at $\infty$ in black for three parameter values: $(A, J) = (1.150,0.3)$ (fig. 2a); $(A, J) = (1.395,0.3)$ (fig. 2b); and $(A, J) = (1.405,0.3)$ (fig. 2c). This figure was obtained by starting with a grid of $640 \times 640$ initial conditions. Initial conditions in the basin of attraction of infinity were determined by seeing which ones yield orbits with large $x$ and $y$ values after a large number of iterations. Those which did are plotted as dots. These dots are so dense that they fill up the black region in the figure. For the case of fig. 2a the boundary of the basin of infinity (i.e., the black region) is apparently a smooth curve (the stable manifold of the period-1 saddle). All initial conditions in the white region generate orbits which remain bounded and are generally asymptotic to the fixed point attractor labeled in the figure. For the case of fig. 2b the basin boundary shows considerably more structure, and, in fact, magnifications of it reveal that it is fractal (cf. refs. 1–4 for discussion of fractal basin boundaries). Initial conditions in the white region are generally asymptotic to a period two attractor (labeled by two dots in the figure) which results from a period doubling bifurcation of the fig. 2a fixed point attractor.*

One of our purposes in this paper will be to describe how a smooth basin boundary, as in fig. 2a, can become a fractal basin boundary, as in fig. 2b, as a system parameter is varied continuously (e.g., as $A$ is varied from 1.150 to 1.395 with $J$ fixed at 0.3). It is found that the boundary becomes fractal following a homoclinic tangency of the period one saddle. Furthermore, we find that this is preceded by a sequence of heteroclinic intersections of stable and unstable manifolds of an infinite number of unstable periodic saddles (section 3). It will be shown that this sequence of events, preceding the actual metamorphosis, determines to a large extent the boundary structure following the metamorphosis to fractal. Henceforth, we shall denote by $A_{sf}$ the value of $A$ for which the boundary becomes fractal as $A$ increases through $A_{sf}$, and we shall call the accompanying conversion a smooth-fractal basin boundary metamorphosis. We conjecture that the dimension of the boundary is greater than one following $A_{sf}$.

We also find another type of basin boundary metamorphosis in which the extent of a fractal basin can grow discontinuously by suddenly sending a Cantor set of thin fingers into the territory of another basin. This is illustrated by a comparison of figs. 2b and 2c which have slightly different values of the parameter $A$ and the same $J$ value: $A=1.395$ in fig. 2b and $A=1.405$ in fig. 2c. Comparing these two figures we see that the basin

---

*The "basin boundary" under study is usually the boundary of the basin of the attractor at infinity. Note that saddle-node bifurcations in the white region do occur and can lead to the presence of more than one attractor there. The white region would be the union of the basins of the attractors not at infinity. Most commonly, however, additional attractors created by saddle-node bifurcations are only seen over a comparatively small range in $A$ before they are destroyed by a crisis [1] (for example, see section 3.1). Thus it is common for the white region to be the basin of a single attractor. We have not observed any saddle-node bifurcations occurring in the black region.
of attraction of infinity (the black region) has enlarged by the addition of a set of thin filaments, some of which appear well within the interior of the white region of fig. 2b (in particular, note the region \(-1.0 \leq x \leq -0.3, 2.0 \leq y \leq 5.0\) in fig. 2c). We call the change exemplified by the transition from fig. 2b to fig. 2c a \textit{fractal-fractal basin boundary metamorphosis}, and we denote the value of \(A\) at which it occurs \(A_{III}\). As \(A\) is decreased toward \(A_{III}\), the filaments become even thinner, their area in the frame of the figure going to zero, but they remain in position, \textit{not} contracting to the position of the basin boundary shown in fig. 2b. Hence, the position of the boundary changes discontinuously although the \textit{area} of the white and black regions apparently changes continuously. (This type of discontinuous jump in the boundary also occurs at \(A = A_{II}\).)

To conclude this subsection we note that Moon and Li [4], in work done at the same time as ours,
have also noted that basin boundaries can become fractal as a result of a homoclinic crossing of stable and unstable manifolds, and they have demonstrated how Melnikov's criterion can be used to find the smooth--fractal metamorphosis condition for a periodically forced Duffing equation. Our analysis, on the other hand, reveals several basic underlying aspects not discussed by Moon and Li [4]: the essential role of accessible orbits, the fractal--fractal metamorphosis, jumps in the location of the boundary at metamorphoses, crisis transfers, and the phenomena discussed in section 3.

1.2. Preliminary discussion of accessible boundary points

Both types of metamorphoses lead to a change in the accessible saddle orbits on the basin boundary. The concept of accessible boundary points and its importance for understanding the structure of fractal basin boundaries will be discussed in section 2.

Definition. A boundary point p is accessible from a region if there is a curve of finite length connecting p to a point in the interior of the region such that no point of the curve lies in the boundary except for p.

As we will describe later (sections 2 and 3), we find that

i) For $A_1 < A < A_{sf}$, the set of boundary points accessible from the white region is the saddle fixed point and its stable manifold (and this is the entire boundary).

ii) For $A_{sf} < A < A_{ff}$, the set of boundary points accessible from the white region is a period-4 saddle and its stable manifold. (No other points are accessible. The stable manifold is dense in the boundary but most boundary points are not accessible.)

iii) For $A > A_{ff}$, the set of boundary points accessible from the white region is a period-3 saddle and its stable manifold*. (No other points are accessible. The stable manifold is dense in the boundary but most boundary points are not accessible.)

In particular, the transition from (i) to (ii) is considered in section 3 and leads us to a study of the macroscopic collision processes that take place as a system parameter is varied. Considering this transition, as well as the transition from (ii) to (iii), we find that a nonattracting saddle set lying in the closure of the basin grows via a complex chain of crossings of stable and unstable manifolds. (This saddle set is a saddle plus the closure of all the crossing points of the stable and unstable manifolds of that saddle.) Eventually this inflated saddle set collides with the boundary, resulting in the basin boundary metamorphosis described above. Alternatively, the saddle set may collide with the attractor (an interior crisis). These virtually invisible saddle sets play a key role in the observed metamorphoses.

In the appendix we describe our numerical techniques for finding accessible boundary saddle orbits.

1.3. Crises transfers

One reason for studying basin boundary metamorphoses is that, as a system parameter is varied, a chaotic attractor can be destroyed by colliding with an unstable orbit on its basin boundary (a boundary crisis [1]). Basin boundary metamorphoses affect boundary crises. We find that basin boundary metamorphoses can lead to changes in the type of crisis which ultimately destroys the attractor, and we call this phenomenon a crisis transfer.

To limit the scope of the problem of describing these metamorphoses we examine a particularly simple scenario. A typical saddle-node bifurcation

* The filaments mentioned above are the stable manifold of the period-3 saddle. The period-3 saddle came into existence much earlier (namely at $A \approx 1.16$) but, before $A_{ff}$, it was in the interior of the white region.
creates a new attractor (namely the node), and the saddle is on the basin boundary. Often the attractor takes a period-doubling route to becoming chaotic. The chaotic attractor then grows as a parameter is varied, collides with the boundary (a boundary crisis), and the attractor and its basin are suddenly destroyed. We wish to describe the major basin boundary metamorphoses that are seen in this scenario, from saddle-node bifurcation to the final boundary crisis.

If one considers the one-dimensional quadratic map (i.e., eqs. (1) with \( J = 0 \)), then it is well known that the attractor in \( |x| < \infty \) no longer exists for \( A > 2 \). What happens is that, as \( A \) increases, the chaotic attractor widens, until, at \( A = 2 \), it touches the unstable period one point on the basin boundary (a boundary crisis [1]). For \( J \), a nonzero fixed but small value, the \( J = 0 \) boundary crisis which occurs as \( A \) is increased is qualitatively unchanged. But for \( J \) larger a qualitative change does occur. This is illustrated in figs. 3a and 3b. These figures are obtained at fixed values of \( J( J = 0.05 \) in fig. 3a and \( J = 0.30 \) in fig. 3b) by plotting a large number of consecutive \( x \) coordinates for an orbit generated by the map, eqs. (1), as the parameter \( A \) is raised in small increments (vertical axis). For figs. 3 the map is iterated \( 10^3 \) times for each value of \( A \). Fig. 3a shows two clear crisis events: one, an interior crisis, at \( A = A^i \approx 1.78 \), in which the chaotic attractor suddenly widens due to a collision with a period-3 saddle orbit which is inside the attractor's basin; and another, a boundary crisis at \( A = A^b \approx 1.8874 \), in which the chaotic attractor is destroyed due to a collision with a saddle fixed point on the basin boundary. The crisis at \( A = A^b \) is essentially the same event as occurs for \( J = 0 \) and \( A = 2 \) in the 1D quadratic map. As \( J \) is raised above the value 0.05 (which applies to fig. 3a), the values \( A^i \) and \( A^b \) move closer together and eventually a crisis transfer occurs. The result is that the final boundary crisis killing the chaotic attractor becomes a period-3 crisis, in which the attractor collides with the unstable period three orbit, which is now on the basin boundary. Figs. 4a and 4b show the chaotic attractor, its basin boundary, and the boundary saddle with which it collides (labeled by arrows) for a case (\( J = 0.05 \), \( A = A^i \)) in which the collision is with a period-1 saddle (fig. 4a) and for a case (\( J = 0.3 \), \( A = A^b \)) in which the collision is with a period-3 saddle (fig. 4b). As the above discussion implies, for \( J = 0.05 \), the period-3 sad-
Fig. 4. (a) The chaotic attractor and its basin at the crisis, $A = 1.8874$, for $J = 0.05$. The period-1 saddle on the basin boundary is indicated by an arrow. (b) The chaotic attractor and its basin at the crisis, $A = 2.12467$, for $J = 0.3$. The elements of the period-3 saddle on the basin boundary are indicated with arrows.

die is not on the basin boundary, but, for $J = 0.3$, it is.

2. Accessible points

2.1. Accessible points for the Hénon map

As mentioned earlier, a point $p$ on the basin boundary is accessible from the white (black) region if one can draw a finite length curve from a point in the interior of the white (black) region to $p$ in such a way that the curve touches the boundary only at that one accessible boundary point. As an illustration, consider the stable and unstable manifolds of the period-1 saddle after they have crossed, i.e., $A > A^*_1$ where $A^*_1$ denotes the value of $A$ at which these manifolds become tangent. We say

$1u \times 1s$

holds when the period-1 unstable manifold crosses the period-1 stable manifold. (The test will make it clear which saddle is under discussion and also which branch of an unstable manifold is under investigation.) A schematic illustration is given in fig. 5. This figure shows that, as a result of the homoclinic intersection, $1u \times 1s$, a series of progressively longer and thinner tongues of the basin of the attractor at infinity (black regions in fig. 2b) accumulate on the outer portion of the stable manifold through the period-1 saddle. [The $(n + 1)$st tongue is the preimage of the $n$th tongue.] A finite length curve connecting a point in the interior of the white region and the period-1 saddle would have to circumvent all the tongues accumulating on the stable manifold of the period-1 saddle. This, however, is not possible because the length of the $n$th tongue approaches infinity as $n \to \infty$. Thus the period-1 saddle is not accessible from the white region. By contrast the period-1 saddle is always accessible from the black region.

This behavior is evident from fig. 2b. In fig. 2b we have also labeled a period-4 saddle orbit. Evidently this saddle lies on the basin boundary and is accessible from the white region. We also note that there are tongues accumulating on the stable manifold of the period-4 saddle but they do so on the side away from the finite attractor (cf. fig. 6a).
After the metamorphosis, however, the boundary includes the period-4 saddle. In this sense we can say that as $A$ increases past $A^*_1$ the basin boundary suddenly jumps inward into the white region. Note, however, that the area of the basin of infinity in any finite region of the plane, numerically, at least, appears to change smoothly as $A$ increases through $A^*_1$.

As soon as $1 u \times 1 s$ holds, the basin boundary is fractal, i.e.

$$A^*_1 = A_{sf}.$$  

Indeed, the fractal dimension of the boundary for the case shown in fig. 2b has been numerically determined to be $d = 1.530 \pm 0.006$. We have done this by using the numerical technique of “uncertainty exponent measurement” introduced in ref. 3.

Next consider the transition from fig. 2b to fig. 2c. We find that the essential difference between these two figures is that, for the parameters of fig. 2b, the accessible saddle is the period-4, while for fig. 2c the period-4 saddle is no longer accessible from the finite attractor, but a period-3 attractor is, as indicated in fig. 2c. Furthermore, the transition between the two cases occurs instantaneously at $A = A^*_4$ when the stable and unstable manifolds of the period-4 saddle are tangent*. Thus

$$A^*_4 = A_{ff}.$$  

Fig. 7 shows a schematic illustration of the tangency of the numerically determined period-4 stable and unstable manifolds at $A = A^*_4 \approx 1.396$.

---

*One branch of the unstable manifold of the period-4 saddles slices through the basin boundary at a Cantor set of points. Here, we are interested in the other branch of the unstable manifold (i.e., the branch which is entirely in the white region in fig. 2b). When it becomes tangent and then crosses the stable manifold of the period-4 saddle, the period-4 saddle will no longer be accessible.
As expected, this value is between those applying for figs. 2b and 2c. Fig. 8a shows a blow-up around an element of the period-3 saddle illustrating that it is accessible. Fig. 8b shows blow-ups of the basin structure in a small region around one element of the period-4 saddle for a case where $A > A_4^*$. The period-4 saddle, which was accessible for $A < A_4^*$ (fig. 6a), no longer is when $A > A_4^*$ (fig. 8b).

2.2. A simple example

In this subsection we describe a model one-dimensional map, $x_{n+1} = F(x_n)$, which illustrates the concept of accessible boundary points in a particularly simple way. The specific map* which

*This map is similar to the one-dimensional version of a two-dimensional map treated in ref. 2 (i.e., set $J = 0$ in ref. 2), and is similar to 1D maps treated by Mira [4], and Takesue and Kaneko [4].
we investigate is piecewise linear and is illustrated in fig. 9a. This map has only two attractors, namely, \( x = + \infty \) and \( x = - \infty \). In addition, as we shall show, it also has the following additional properties:

i) There is a Cantor set basin boundary separating the basins of the two attractors.

ii) The only periodic orbit on the basin boundary which is accessible from the attractor basin of \( x = + \infty \) (\( x = - \infty \)) is the unstable fixed point at \( x = +1 \) (\( x = -1 \)).

iii) Each boundary point which is accessible from the \( x = + \infty \) (\( x = - \infty \)) basin is mapped to \( x = +1 \) (\( x = -1 \)) in a finite number of iterates.

iv) The Cantor set basin boundary contains an infinite number of unstable periodic orbits, and, except for the points \( x = \pm 1 \), all of these are inaccessible from either basin.

To demonstrate (i)-(iv) we first note that, for any point \( x_n > 1 \), the map may be expressed

\[
(x_{n+1} - 1) = 5(x_n - 1).
\]

Thus any initial condition in \( x > 1 \) generates an orbit which tends to \( + \infty \), and \( x > 1 \) is part of the basin for the \( x = + \infty \) attractor. By symmetry, \( x < -1 \) is, similarly, part of the basin of the \( x = - \infty \) attractor. From fig. 9a we also see that the points \( 0.2 < x < 0.6 \) (\( -0.6 < x < -0.2 \)) map on one iterate into \( x < -1 \) (\( x > +1 \)). Hence, \( 0.2 < x < 0.6 \) is in the \( - \infty \) basin, and \( -0.6 < x < -0.2 \) is in the \( + \infty \) basin. This is illustrated in fig. 9b [the symbol \( 1 + (1-) \) signifies that the designated interval maps to \( x > +1 \) (\( x < -1 \)) in \( n \) iterate of the map]. Continuing the construction, one finds three open intervals which map to \( l + \) in one iterate and hence to \( x > 1 \) in two iterates (these intervals are labeled 2 + in fig. 9c), and three more open intervals which map to \( x < -1 \) in two iterates (labeled 2 -). The \( 2 + (2-) \) intervals are in the basin of \( + \infty (- \infty) \). We see that the basin boundary must lie in \([-1, +1]\) and does not contain any of the eight intervals labeled \( 1 + , 1 - , 2 + , 2 - \). This is clearly the first two stages in a standard Cantor set construction; this demonstrates (i). [The fractal dimension of this Cantor set basin boundary is \((\ln 3)/\ln 5) = 0.6826\ldots\). Further, we see that the boundary points accessible from \( 1 + (1-) \) are \( x = -0.6 \) and \( x = -0.2 \) (\( x = 0.6 \) and \( x = 0.2 \)), and these map in one iterate to \( x = +1 \) \( (x = -1) \). Similarly, the boundary points accessible from the \( 2 + (2- \) intervals are seen to all map to \( x = +1 \) \( (x = -1) \) in two iterates. Continuation of the Cantor set construction implies point (iii) and hence point (ii). Since the map takes three subintervals of \([-1, +1]\) and linearly stretches them into the whole interval (the subintervals are \([-1, -0.6], [-0.2, +0.2], \text{ and } [0.6, 1] \), there are an infinite number of unstable periodic orbits* in \([-1, +1]\). These points cannot lie in either basin and hence must lie on their boundary. However, by (iii) they are also not accessible. Hence (iv) follows.

*To see this, note that each of the three subintervals themselves have three subintervals (cf. fig. 9c) which are mapped onto \([-1, +1]\) in two iterates, and so on. Thus, the \( n \) times iterated map has \( 3^n \) subintervals which are linearly stretched and mapped to \([-1, +1]\). Each of these thus contains a fixed point of the \( n \) times iterated map. Hence, there are \( 3^n/n \) periodic orbits of period \( n \).
As an example of an inaccessible boundary point, consider the fixed point \( x = 0 \). From the construction of the Cantor set, we see that this point certainly is on the boundary but that any \( \epsilon \) neighborhood of it contains an infinity of alternating segments of the two basins. [Note that the accessible points are those points which are end points of one of the intervals deleted as one constructs the Cantor set. In addition, they are precisely those points that map exactly onto one or the other fixed points at \( x = \pm 1 \). Such points could be considered the (generalized) stable manifold of the fixed points.]

The example just given is analogous to the fractal basin boundary in fig. 2b in the following ways. First, a straight line cutting across the basin...
boundary of fig. 2b intersects it in a Cantor set along that line. Second, there are two accessible periodic orbits in the boundary in both cases: the period one saddle and the period four saddle, for fig. 2b; and the two unstable fixed points \( x = \pm 1 \), for the one-dimensional map model. Third, all points accessible from the white (black) region of fig. 2b are on the stable manifold of the period four (one saddle); while all accessible points of the \( +\infty(-\infty) \) attractor for the one-dimensional map model map to \( x = +1(x = -1) \) in a finite number of iterates. Fourth, both boundaries contain an infinite number of inaccessible unstable periodic orbits, which, in the case of fig. 2b, are saddles. Section 3 will, in fact, show how an infinite number of saddles get buried in the boundary as a result of the basin boundary metamorphosis.

Now we consider how the one dimensional map with a fractal basin boundary, given in fig. 9, might come about with variation of a system parameter. That is, we consider a basin boundary metamorphosis for a one-dimensional noninvertible map. As shown in fig. 10, we imagine that as some parameter \( p \) is raised from \( p_1 \) to \( p_2 \) the lower minimum of the map moves from \( F(x) > -1 \) at \( p = p_1 \) [fig. 10a] to \( F(x) = -1 \) at \( p = p_{st} \) [fig. 10b] to \( F(x) < -1 \) at \( p = p_2 \) [fig. 10c]. For \( p = p_1 \) the basin boundary is simply the single point \( x = 1 \), but there is also an invariant Cantor set in \( 1 > x > -1 \) (i.e., entirely in the \( +\infty \) basin) generated due to the fact that the positive maximum of \( F(x) \) is greater than one. For fig. 10c there is a fractal basin boundary (as in fig. 9), and the conversion to fractal occurs at \( p = p_{st} \). When this occurs the basin boundary dimension changes discontinuously. In particular, it is zero for \( p < p_{st} \) (the boundary is the single point \( x = 1 \)), but as \( p \) approaches \( p_{st} \) from above it is the dimension of the invariant Cantor set in the limit as \( p \) approaches \( p_{st} \) from below. Furthermore, as \( p \) increases through \( p_{st} \) the basin boundary jumps far into the interior of the \( +\infty \) basin. Indeed, for \( p \) just slightly bigger than \( p_{st} \), there are thin interval pieces of the \( -\infty \) basin near all the elements of the Cantor set which, for \( p < p_{st} \), existed entirely in the \( +\infty \) basin. Just as the thin filaments which jump into the white basin in fig. 2c approach zero width as \( \Delta \to \Delta_{st} \) from above, so too do the above-mentioned interval pieces of the \( -\infty \) basin approach zero as \( p \to p_{st} \) from above.

From the point of view of directly observable consequences, perhaps the most significant aspect of the basin boundary structure we have described here in section 2 is its implications for the final crisis in which the chaotic attractor is destroyed as it collides with a saddle on the basin boundary. Although there are an infinite number of saddles embedded in the basin boundary, the attractor can collide only with that one which is accessible to it. Thus, the type of crisis is, in a sense, determined by the accessible boundary saddle at the crisis (cf. figs. 4). Further, a crisis transfer may be thought
3. How saddles get buried in the basin boundary and how the boundary jumps inward

It is evident from the discussion of figs. 2 in section 2.1 that the two events occurring at $A = A_{sf}$ and at $A = A_{fr}$ are closely analogous. Both involve homoclinic crossings of accessible boundary saddle orbits: $1u \times 1s$ at $A_{sf}$ and $4u \times 4s$ at $A_{fr}$. (At those values the relevant stable and unstable manifolds are tangent.) Also, both transitions are accompanied by a jump of the basin boundary inward into the white region. For the case of the smooth-fractal basin boundary metamorphosis, the inward jump occurs because the period-4 saddle is a finite distance from the basin boundary when $A < A_{sf}$, while for $A > A_{sf}$ there are parts of the basin of the attractor at infinity which are arbitrarily close to the period-4 saddle. A similar statement applies for the period-3 saddle at the fractal-fractal basin boundary metamorphosis.

With this in mind, we study the events leading up to the metamorphosis at $A_{fr}$ (since $A_{fr}$ is similar). In particular, we are interested in determining why the basin boundary suddenly jumps inward to the period-4 saddle at the onset of $1u \times 1s$, and in determining how an infinite number of other inaccessible saddles get buried in the basin boundary. We do not have a complete understanding of the answers to these questions. For example, we shall show how a particular class of saddles get buried in the basin boundary. We call saddles in this class simple Newhouse saddles*. While there are an infinite number of simple Newhouse saddles they are a comparatively small subset of all the buried saddles. Nevertheless, our study will be sufficient for understanding why the basin boundary jumps inward at $A_{sf}$.

\*Newhouse [5] was one of the first to give an extensive picture of how saddle-node bifurcations occur near homoclinic tangencies, and he developed an extensive theory of them. See also references in [6].
3.1. Simple Newhouse orbits

We begin by describing in more detail the sequence of events for $J = 0.3$ as $A$ is increased. As $A$ increases a saddle-node bifurcation occurs at $A_1 = -(1 + J)^2/4 = -0.4225$ in which a period-1 attractor and a period-1 saddle are created. In addition, an infinite number of other saddle-node bifurcations of other higher periods also occur for larger $A$. For these higher period orbits the stable orbit created is generally observed to proceed rapidly (with increase of $A$) through a period-doubling cascade to a final crisis that destroys the attractor and its basin. The remnant of such a sequence following a period-$N$ saddle-node bifurcation (i.e., what is left following the final crisis) is a horseshoe (or at least it rapidly becomes a horseshoe) containing an infinite number of saddle orbits. In particular, one of these unstable orbits will be the period-$N$ saddle created at the original saddle-node bifurcation. We call these saddles original saddles. These saddles have the property that their associated eigenvalues of the linearized map are both positive (since $J > 0$). Furthermore, we shall concentrate our discussion on a special class of these original saddles. In order to delineate this class, consider fig. 11a which schematically illustrates the stable and unstable manifolds of the original period-1 saddle for a value of $A$ below that at which the manifolds are tangent. As $A$ increases, a homoclinic tangency, as shown in fig. 11b, occurs at some value of $A$ which we denoted $A^*$. (This value is also the value after which the basin boundary of infinity becomes fractal, $A^* = A_{fr}$. ) For now, however, we wish to discuss the occurrence of saddle-node bifurcations (and hence the creation of original saddles) for $A < A^*$ as $A \rightarrow A^*$. In particular, we observe an infinite sequence of saddle-node bifurcations of period $3, 4, 5, 6, 7, \ldots$, which occur at parameter values $A_3, A_4, A_5, \ldots$, where $A_3 < A_4 < A_5 < A_6 < A_7 < A_8 \cdots < A^*$. The particular bifurcations which we refer to above are those which create original saddles that proceed once around a circuit following the unstable manifold, as schematically illustrated in fig. 12 for a period-5 case. We call these orbits simple Newhouse orbits*. It can be shown [5] that for large $n$

$$A^* - A_{fr} \sim \lambda^{-n}.$$ 

*Actually [5, 6] there is a much larger class of saddle-node bifurcations than the ones we concentrate on here. For example, in general, a high period orbit may proceed several times around the loop before returning to its original value.
and, furthermore, the width $\Delta_n$ of the window is [6]

$$\Delta_n \sim \lambda^{-2n},$$

where $\lambda > 1$ is the unstable eigenvalue at the original period-1 saddle for $A = A^*_1$, and $\Delta_n$ denotes the difference between $A_n$ and the value of $A$ at the final crisis of the attractor created at $A_n$. Thus we see that, for large $n$, the range of $A$ over which the attractor exists is small compared to the range of $A$ between the initial period-$n$ saddle-node bifurcations and the homoclinic tangency of the period-1 saddle manifolds.

### 3.2. Crossing of the stable manifold of simple Newhouse orbits with the unstable period-1 manifold

For example, the saddle-node bifurcation of the simple Newhouse orbit of period-3 occurs at $A \approx 1.16$ and the final crisis of the resulting chaotic attractor (which consists of three pieces) occurs at $A \approx 1.25$. Fig. 13a shows a schematic diagram of the basin of attraction for the period-3 attractor at $A = 1.25$. The actual numerically determined period-3 basin appears in fig. 13b. Also shown in fig. 13a are the basin of attraction for the period-1 attractor, the unstable manifold of the original period-1 saddle, and the location of the period-3 saddle. The basin boundary for the period-3 at-
tractor is the stable manifold of the period-3 original saddle, and the basin boundary of infinity is the stable manifold of the period-1 saddle. When the period-3 saddle-node bifurcation occurs, it immediately creates a basin for itself by cutting a chunk out of the period-1 basin of attraction*. It is important to note from fig. 13a that the stable manifold of the period-3 saddle (the basin boundary for the period-3 attractor) cuts across the unstable manifold of the period-1 saddle. In fact, this is true for all values of $A$ for which the period-3 saddle exists, including those exceeding the crisis value for the three-piece chaotic attractor (which evolves via a period doubling cascade from the period-3 node). In this range, $A > 1.25$, of course, the stable manifold of the period-3 saddle is no longer a basin boundary. (In fact, we shall be most interested in what happens near $A < 1.315$).

Furthermore, we observe an analogous situation to hold for all simple Newhouse orbits. That is, for any simple Newhouse orbit the stable manifold always cuts across the period-1 saddle’s unstable manifold. (While at $A = 1.25$ the period-3 saddle has one branch of its unstable manifold becoming tangent and crossing its stable manifold, we are interested in what the other unstable branch does (cf. next subsection.).)

### 3.3. Crossing of the unstable period-$N$ manifold with the stable period-$N + 1$ manifold

We have numerically examined the stable and unstable manifolds of the simple Newhouse orbits, and we have observed the following sequence of events:

1) After $A = A_{4,5}^* = 1.308$ the unstable manifold of the period-4 saddle crosses the stable manifold of the period-5 saddle.

2) After $A = A_{5,6}^* = 1.310$ the unstable manifold of the period-5 saddle crosses the stable manifold of the period-6 saddle.

3) After $A = A_{6,7}^* = 1.311$ the unstable manifold of the period-6 saddle crosses the stable manifold of the period-7 saddle.

Fig. 14 shows the crossings of the numerically determined period-4 unstable ($4u$) and period-5 stable ($5s$) manifolds. One can argue rigorously that an infinite number of such crossings $A_{n,n+1}^*$ occur as $A \to A_{41}^*$:

$$
\begin{align*}
4u \times 5s \\
5u \times 6s \\
6u \times 7s \\
7u \times 8s \\
8u \times 9s \\
& \vdots \\
Nu \times (N + 1)s \\
& \vdots
\end{align*}
$$

where $Nu$ and $Ns$ stand for the period-$N$ unstable (stable) manifold, and $\times$ signifies crossing of the two relevant manifolds*.

### 3.4. Heteroclinic manifold crossings

We now review the significance of the crossing of a stable and an unstable manifold when each are associated with different saddle periodic orbits. Consider two saddles, one of period $M$ and one of period $N$. Say, that as a parameter of the system is varied the unstable manifold of the period-$M$ saddle crosses the stable manifold of the period-$N$ saddle. Also assume that the eigenvalues associated with these saddles are positive (as for simple Newhouse orbits). The situation is as illustrated in fig. 15. Evidently the unstable manifold of the

*Note, however, that the period-3 saddle does not appear in the sequence (3). Its crossing, $3u \times 4s$, occurs later, immediately following $A = A_{4,5}^* = A_{41}^* > A_{41}^*$. This is implied by analogy with the $1u \times 1s$ event at $A_{11}^*$ for which we show (section 3.5) that the $1u \times 1s$ and $4u \times 1s$ events are simultaneous.
Fig. 14. Numerically obtained period-4 unstable manifold and period-5 stable manifold at $A = 1.3087$, $J = 0.3$.

Fig. 15. Heteroclinic crossings for a period-$M$ saddle orbit and a period-$N$ saddle orbit. Only one point of each periodic orbit is shown.

Period-$M$ saddle limits on the period-$N$ unstable manifold, and the period-$N$ stable manifold limits along the entire period-$M$ stable manifold. Thus the closure of the period-$M$ unstable manifold contains the period-$N$ unstable manifold, and the closure of the period-$N$ stable manifold contains the period-$M$ stable manifold,

\[
\overline{M_u} \supseteq N_u, \quad \overline{N_s} \supseteq M_s,
\]

where the overbar denotes closure. In statements like (4), we write $N_u(N_s)$ to denote the unstable (stable) manifold (both branches included) of the simple Newhouse saddle of period $N$.

Eqs. (4) have strong implications for the structure discussed in sections 3.2 and 3.3. According to section 3.2, the stable manifold of any simple Newhouse orbit crosses the unstable manifold of the period-1 saddle. Thus, from (4),

\[
\overline{t_u} \supseteq N_u, \quad \overline{N_s} \supseteq 1_s
\]

for any period-$N$ simple Newhouse orbit. Similarly sequence (3) of section 3.3 implies that as $A \to A_{sf}$ from below, we have

\[
\begin{align*}
\overline{4u} & \supseteq 5u \supseteq 6u \supseteq \cdots, \\
\overline{4s} & \subseteq 5s \subseteq 6s \subseteq \cdots.
\end{align*}
\]

Combining (5) and (6) we have

\[
\begin{align*}
\overline{t_u} & \supseteq \overline{4u} \supseteq 5u \supseteq 6u \supseteq \cdots, \\
\overline{1s} & \subseteq \overline{4s} \subseteq 5s \subseteq 6s \subseteq \cdots,
\end{align*}
\]

for $A \geq A^*_1$. Thus, for example, eq. (7a) implies that the unstable manifold of the period-1 saddle is extremely complex and convoluted, since its closure includes all the unstable manifolds of all simple Newhouse orbits.

3.5. Homoclinic crossing for the period-1 saddle

We now claim that when the stable and unstable manifolds of the period-1 saddle cross, then simultaneously the $4u, 5u, \ldots$ manifolds also cross the period-1 stable manifold. That is, the relationships $1s \times 1u$ and $1s \times N_u$, $N \geq 4$, are valid immediately following $A = A^*_1$.

In order to see how this arises consider, for example, the period-4 unstable manifold. By eq. (7a), and as illustrated in fig. 16, $4u$ comes arbitrarily close to all the simple Newhouse orbits of period $5, 6, \ldots$. As $A \to A^*_1$ an infinite number of such orbits are created by saddle node bifurcations. The locations of these orbits are closer and
closer to the stable manifold of the period-1 saddle as the period of these orbits gets higher and higher. In fact, in a suitable sense we can regard the tangency points of $1_s$ and $1_u$ at $A = A_i^*$ (cf. fig. 11b) as the limit of the elements of a simple Newhouse saddle of period $N$ as $N \to \infty$ (cf. ref. 6). Thus, as $A \to A_i^*$, the closure of the period-4 unstable manifold touches the period-1 stable manifold. As soon as $A$ exceeds $A_i^*$, a set of tongues, as in fig. 5, come shooting up along the formerly smooth basin boundary. These tongues must cross the period-4 unstable manifold. Hence, $1_s \times 1_u$ and $1_s \times 4_u$ become true simultaneously and similarly for $1_s \times N_u$, $N > 4$.

Since the $1_s \times N_u$, $N \geq 4$ relationships become true simultaneously, we see that eq. (4) implies that

$$N_u \supseteq 1_u \quad \text{and} \quad N_s \subseteq 1_s$$

(8)

for $A > A_i^*$, $N \geq 4$. (Note that the period-3 simple Newhouse orbit does not experience this type of event at $A = 1.315$.) Eqs. (8) and (7) imply that for $A > A_i^* = 1.315$ we have for the simple Newhouse orbits

$$1_u = 4_u = 5_u = 6_u = \cdots,$$  \hspace{1cm} (9a)
$$1_s = 4_s = 5_s = 6_s = \cdots.$$  \hspace{1cm} (9b)

Thus at the smooth–fractal basin boundary bifurcation the closures of all the simple Newhouse unstable manifolds become equal and similarly for their stable manifolds. In particular, the basin boundary for the attractor at infinity will be

$$\text{Basin boundary} = 1_s = 4_s.$$

(9c)

Since the period-1 and the period-4 saddles are the only accessible saddles, all the other simple Newhouse orbits ($N > 4$) are "buried" in the boundary.

3.6. Basin boundary jumps

Thus we see from eqs. (9) that all the simple Newhouse orbits of period greater than or equal to 4 lie in the basin boundary. Furthermore, equalities (9a) and (9b) imply a striking discontinuity in the boundary at $A_{sf}$. The period-4 saddle is certainly not on nor is it especially near the boundary at or before $A_{sf}$; yet we see it is part of the boundary immediately after $A_{sf}$. Since this orbit necessarily moves smoothly and continuously, it follows that the boundary discontinuously jumps inward to the orbit. This also holds for the period-5 (simple Newhouse) saddle and period 6, etc. But the lower the period is, the more striking the conclusion since the orbits of lower period are further from the boundary. For other values of $J$, the period 4 must be replaced by some appropriate period and the accessible orbit might not be a simple Newhouse orbit. Based upon the ideas developed in this paper, the following rigorous result [6] has been obtained for the smooth fractal transition:

**Theorem.** Consider a typical map with a saddle fixed point or periodic orbit $P$ that has a transition value $A_{sf}$ as $A$ increases (where the stable and unstable manifolds of $P$ cross for the first time). Assume the absolute value of the determinant of the Jacobian at $P$ is less than one (that is, the Jacobian of the $N$th iterate of the map where $N$ is the period of $P$). Then there will be a periodic saddle $Q$ that is in the closure of the stable manifold of $P$ for all $A$ slightly greater than $A_{sf}$ but is not in it at $A_{sf}$. Furthermore, $P$ will be in
the closure of the stable manifold of $Q$ for all $A$ slightly greater than $A_{sl}$.

S. Hammel and C. Jones [7] were the first to prove a result with these hypotheses concluding that the basin boundary jumps inward. Their version makes no mention of the orbit $Q$. Their techniques are totally different from the ideas in eqs. (9), though the ideas in eqs. (9) are a sufficient foundation to give a rigorous proof of this theorem.

In addition, note that this theorem also applies to the $A_{ff}$ transitions, since for the period-4 saddle the first crossing of its right-hand (inward) unstable manifold with its stable manifold follows $A_{ff}$.

4. Summary

In this paper [8] we have investigated sudden large scale changes in basin boundaries with variation of a system parameter. Our conclusions are as follows:

i) The basin boundary can jump in size and change its character as the system parameter passes through certain critical values, and we call these changes basin boundary metamorphoses.

ii) Metamorphoses can occur as a result of homoclinic intersections of the stable and unstable manifolds of a saddle periodic orbit on the basin boundary.

iii) The structure of the basin boundaries which we investigate is, to a large extent, determined by the accessible saddles which lie on the basin boundary (section 2).

iv) The character changes referred to in (i) are changes in the accessible saddle orbits on the basin boundary and sometimes a change of the boundary from being smooth to being fractal (as in the transition from fig. 2a to fig. 2b but not from fig. 2b to fig. 2c).

v) The way in which saddles get buried in the basin boundary (thereby becoming inaccessible) and the means by which basin boundaries can jump in size have been investigated, and it has been shown that these are accomplished by a complex sequence of events which precede the basin boundary metamorphosis (section 3). In particular, a saddle set lying in the basin grows via a chain of crossings of stable and unstable manifolds. Eventually this inflated saddle set collides with the boundary, resulting in the metamorphosis described above*.

vi) The saddle which a chaotic attractor collides with at a boundary crisis is the boundary saddle accessible to that attractor. As a parameter is varied, a change in the accessible saddle at the crisis boundary is called a crisis transfer (section 1.3).

Acknowledgements

We would like to thank Shen-Teng Yang for making the pictures of fig. 1 and Bae-Sig Park for numerically determining the dimension of the basin boundary corresponding to fig. 2b. This work was supported by the Department of Energy (Office of Basic Energy Sciences), DARPA under NIMPP, and the Office of Naval Research.

Appendix

Finding accessible boundary saddles

In fig. 2, we have indicated accessible saddles on the basin boundary. Here we describe our numerical technique for the determination of these accessible periodic boundary saddle orbits. Say we have two attractors, attractor A and attractor B, and we wish to find a boundary saddle orbit that is accessible from the basin of attractor B. We follow the following steps:

1) Choose two points close to the basin boundary that go to the two different attractors, say, point $P_A$ goes to attractor A and $P_B$ goes to attractor B.

*Alternatively, the attractor may collide with this inflated saddle set before this set has collided with the boundary (an interior crisis). This is what is happening at the period three interior crisis of fig. 3a.
2) Take 64 points equally spaced in the segment \( P_A P_B \) and number them sequentially so that \( P_A = 1 \) and \( P_B = 64 \). (We choose 64 points because 64 is the vector length in Cray computers.)

3) Iterate the 64 points and determine the highest numbered point which goes to the attractor \( A \); call this the new point \( P_A^- \).

4) Move the point \( P_B \) a fraction (say, a fifth) of the distance \( P_A^- P_B \) towards \( P_A^- \) and make sure that the new point \( P_B^- \) is in the basin of \( B \). If it is not, move back in small steps (in the direction away from \( P_A^- \)) until it is.

5) Go back to step 2 and keep repeating the procedure until the distance \( P_A^- P_B^- \) is less than, say \( 10^{-12} \).

6) Iterate \( P_A^- \) and \( P_B^- \) and print the first, say, 50 points of the trajectories.

7) Extract the period and the elements of the accessible boundary saddle orbit from the printed trajectories.

For the cases in figs. 2, for different choices of the initial points \( P_A \) and \( P_B \), this procedure always yields the indicated boundary saddles accessible from the black and white regions, and we thus believe that these are the only accessible boundary saddles.

References