Hopf Bifurcation: The Appearance of Virtual Periods in Cases of Resonance*

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1. INTRODUCTION

Hopf's Bifurcation Theorem [7] describes conditions under which a family of nonconstant periodic orbits bifurcates from a path of stationary solutions of a parametrized differential equation. In addition, the theorem provides information about the periods of such orbits in terms of the eigenvalues of the linearized system at the bifurcation point. In this paper we examine the consequences of relaxing certain hypotheses of the Hopf Theorem.

For \((x, \lambda) \in \mathbb{R}^n \times R\), let
\[
\frac{dx}{dt} = f(x, \lambda),
\]
where \(f: \mathbb{R}^n \times R \to \mathbb{R}^n\) is a \(C^1\) function such that \(f(0, \lambda) \equiv 0\); and let \(A(\lambda) = D_x f(0, \lambda)\). We assume that \(A(0)\) is nonsingular. In the following, "orbit" will always refer to a nonconstant periodic orbit, and "period" will always mean the minimum period of such an orbit. Since the eigenvalues of \(A(\lambda)\) vary continuously with \(\lambda\), we can map \(R\) continuously into the set of eigenvalues and (for \(\lambda \in R\)) refer to a path \(\alpha(\lambda) + i\beta(\lambda)\) of eigenvalues of \(A(\lambda)\).

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Suppose that such a path of eigenvalues crosses the imaginary axis at $i\beta$ when $\lambda=0$ (i.e., $\alpha(0)=0$ and $i\beta(0)=i\beta$). Then, under additional assumptions, Hopf proved that a family $\Gamma$ of orbits of (1.1) (in this case, a continuous path of orbits) emanates from $(0,0)$. (See, e.g., [12].) The additional assumptions required by his techniques are

1. $\alpha'(0) \neq 0$ and
2. if $\mu$ is another eigenvalue of $A(0)$, then $\mu/\pm i\beta \neq$ integer.

Hopf proved further that as $\Gamma$ approaches $(0,0)$ the periods of the orbits on $\Gamma$ approach $2\pi/\beta$. In this case, we say that $\Gamma$ bifurcates from $(0,0)$ starting with period $2\pi/\beta$.

Here we investigate the situation in which multiple paths of eigenvalues cross the imaginary axis when $\lambda=0$ and $A(0)$ violates hypothesis (2). Our results imply that generically (i.e., when orbits satisfy certain non-degeneracy conditions) the Hopf Theorem extends to the case in which $\mu/\pm i\beta$ is any integer except 1 or 2. Even in the general case, the results almost extend, except that we must look at "virtual periods" instead of periods. We will define the concept of virtual period after some preliminary discussion.

Assume that $n=4$, and suppose that $A(\lambda)$ has paths of eigenvalues which cross the imaginary axis (with nonzero speed) at $\pm i$ and $\pm i\beta$ (with $\beta \geq 1$) when $\lambda=0$. If $\beta$ is not an integer, then Hopf's theorem implies that two families of orbits bifurcate from $(0,0)$, starting with periods $2\pi$ and $2\pi/\beta$, respectively. However, if $\beta$ is an integer (in which case the system is said to be in resonance) and if $f(0,\cdot)$ is nonlinear, then it is not necessarily the case that two families bifurcate. If $\beta > 1$, then the Hopf Theorem guarantees that one family bifurcates from $(0,0)$, starting with period $2\pi/\beta$. In Section 2, we discuss examples (with $\beta > 1$) in which only one family bifurcates. If $\beta = 1$, then the theorem does not guarantee any bifurcation. (See D. Schmidt [11] for an example of this type of resonance in which no orbits bifurcate.) Here we concentrate on the case $\beta > 1$ and analyze how one of the two expected bifurcating families can disappear.

Consider the analogous but simpler situation of a vector field $v$ on $S^2$. Generically, we expect two (or more) critical points. It is possible, however, to have a single degenerate critical point, $x_0$. As a singularity of $v$, $x_0$ necessarily has index $+2$. Instead of examining the index, however, we could detect the degeneracy of $x_0$ by noting that $Dv(x_0)$ must have $+1$ as an eigenvalue. Our approach is analogous. Just as two generally expected fixed points can be fused into a degenerate one, two generally expected families of periodic orbits can be fused into one. We relate the study of periodic orbits to that of fixed points by using the Poincaré or "first-return" map of an orbit. In the case of orbits, this degeneracy can be detec-
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ted by looking at the eigenvalues of the linearized Poincaré map. Hence, we make the following definition:

Let $T$ be the Poincaré map associated with an orbit $y$ of (1.1) at the point $x_0$. We assume that $T$ is defined on an $(n-1)$-disk $D \subset \mathbb{R}^n$ which is transverse to the orbit. Let $\tau$ be the period of $y$, and let $G = D_x T(x_0)$. If there exists a point $y \in \mathbb{R}^{n-1}$ with $y, Gy, G^2y, \ldots, G^{m-1}y$ distinct, but $G^m y = y$, for some $m \geq 1$, then we say $\tau = m\tau$ is a virtual period of $y$. When $y \neq 0$, we say $m$ is the order of the virtual period. We also say the period is a virtual period of order 0.

Note that if $y$ has a virtual period of order $m$, where $m > 1$, then $G$ will have an eigenvalue which is an $m$th root of unity, for some divisor $r$ of $m$. Since $G$ is linear, $y$ can have only finitely many virtual periods. Thus the virtual periods form a special subset among all multiples of the period of $y$. (The virtual period phenomenon is illustrated in the examples of Sect. 2).

Consider again the four-dimensional case where $A(\lambda)$ has paths of eigenvalues which cross the imaginary axis at $\pm i\beta$ and $\pm i\beta (\beta \geq 1)$. Our main theorem (Theorem 3.6 in Sect. 3) implies that for $\beta \neq 1, \beta \neq 2$, not only will a family of orbits bifurcate from $(0,0)$ starting with period $2\pi/\beta$, but a family will also bifurcate starting with virtual period $2\pi$. In case the families are distinct, the second family will, of course, start with actual period $2\pi$. Again, however, in the case of resonance, it is possible for only one family to bifurcate. In this case, as the family approaches $(0,0)$, the actual periods of the orbits approach $2\pi/\beta$; but the orbits must also have virtual periods which approach $2\pi$. Thus we demonstrate that when a change in the eigenvalue $i\beta$ brings the system into resonance, although the orbits of actual period near $2\pi$ may disappear, orbits on the bifurcating family still exhibit the period $2\pi$ in the form of a virtual period. The cases $k = 1, k = 2$, and the higher dimensional cases require further discussion (see Sect. 3).

In Section 4, we use Theorem 3.6 to obtain a corresponding Center Theorem (Theorem 4.5) involving virtual periods for Hamiltonian systems.

2. Examples

In this section we describe two examples of Hopf bifurcation which will illustrate the concept of virtual period and provide motivation for the main theorem of Section 3. Here we want to consider simple cases of bifurcation in the presence of resonance. In particular, we focus on parametrized systems

$$\dot{x} = f(x, \lambda),$$

where $f \in C^1(\mathbb{R}^4 \times \mathbb{R}, \mathbb{R}^4)$, $f(0, \lambda) = 0$, and $A(\lambda) = D_x f(0, \lambda)$ has paths of eigenvalues which cross the imaginary axis at $\pm i$ and $\pm ki$ ($k = 2, 3, 4, \ldots$)
when \( \lambda = 0 \). In cases such as these, a family \( \Gamma \) of orbits bifurcates from \((0,0)\) starting with period \(2\pi/k\), and one of three situations can occur (see Theorem 3.6):

- (b1) another family of orbits bifurcates from \((0,0)\) starting with period \(2\pi\);
- (b2) the orbits on \( \Gamma \) have virtual periods of order \(k\), and no other families bifurcate from \((0,0)\); or
- (b3) the orbits on \( \Gamma \) have no virtual periods near \(2\pi\) (i.e., no virtual periods of order \(k\)), and no other families bifurcate from \((0,0)\).

In Section 3 we show that (b3) can only occur when \(k = 2\). Viewed together, the following examples illustrate all three possibilities.

**Example 1.** For \((u_1, u_2, v_1, v_2) \in R^4\), let \(\dot{x} = Ax\) \((x \in R^4)\) represent the linear system

\[
\begin{align*}
\dot{u}_1 &= u_2, \\
\dot{u}_2 &= -u_1, \\
\dot{v}_1 &= kv_2, \\
\dot{v}_2 &= -kv_1,
\end{align*}
\]

where \(k\) is a positive integer greater than 1. The eigenvalues of \(A\) are \(\pm i\) and \(\pm ki\). Every point in \(R^4\) lies on a (periodic) orbit of (2.1). Orbits in the \(v\) plane have period \(2\pi/k\); and others have period \(2\pi\). The unit sphere \(S^3\) is invariant under (2.1). In \(S^3\) each orbit lies on a torus determined by the number \(s \in [0, 1]\), where \(u_1^2 + u_2^2 = s\) and \(v_1^2 + v_2^2 = 1 - s\). When \(s = 0\), (resp. \(s = 1\)), the tori degenerate to a single orbit in the \(v\) plane, (resp. \(u\) plane). The orbit \(\gamma\) in the \(v\) plane is the only orbit of period \(2\pi/k\) on \(S^3\). Let \(T\) be the Poincaré map for the orbit \(\gamma\) at the point \(x_0\) on \(\gamma\). Restricted to points in \(S^3\), \(T\) has one fixed point, \(x_0\). All other points are periodic of period \(k\), corresponding to the orbits of period \(2\pi\) around \(\gamma\).

Now we describe a perturbation of (2.1) on \(S^3\) which destroys all closed orbits except \(\gamma\). (Trajectories starting on \(S^3\) will remain on \(S^3\).) If each orbit on \(S^3\) is identified (to a point), the resulting identification space is homeomorphic to \(S^2\). Let \(p_0 \in S^2\) be the image of \(\gamma\) under this projection, and let \(v\) be a \(C^1\) vector field on \(S^2\) with only one zero at \(p_0\). Then the index of \(p_0\) with respect to \(v\) is necessarily 2. We lift \(v\) to a \(C^1\) vector field \(g\) on \(S^3\), perpendicular to the orbits of (2.1). For \(x \in S^3\), the resulting system

\[
\dot{x} = Ax + g(x)
\]
has only one orbit, \( \gamma \). Let \( \overline{\mathcal{T}} \) be the Poincaré map of \( \gamma \) under (2.2). Further analysis shows that the fixed point \( x_0 \) of \( \overline{\mathcal{T}}^k \) has fixed-point index \( k + 1 \). (See Fig. 1.)

Extend (2.2) to \( \mathbb{R}^4 \) by setting

\[
\dot{x} = F(x) = \begin{cases} 
Ax + ||x||^3 \left( \frac{x}{||x||} \right) & \text{for } x \neq 0 \\
0 & \text{for } x = 0.
\end{cases}
\] (2.3)

Note that \( x = 0 \) is the only stationary point of (2.3), \( D_x F(0, 0) = A \), and each sphere centered at \( x = 0 \) is invariant. The sphere \( S_r = \{ x \in \mathbb{R}^4: ||x|| = r \} \) contains exactly one orbit \( \gamma_r \) of (2.3). For each \( r > 0 \), the orbit \( \gamma_r \) has period \( 2\pi/k \) and lies in the \( r \) plane. Let \( \Gamma = \bigcup_{r > 0} \gamma_r \).

Now, we embed (2.3) into a parametrized system. Let

\[
\dot{x} = f(x, \lambda) = F(x) + \lambda x.
\] (2.4)

This system has a path of stationary points at \( x = 0 \) (i.e., the \( \lambda \) axis). For \( \lambda \neq 0 \), it has no orbits.\(^1\) At \( \lambda = 0 \), the family \( \Gamma \) of orbits of (2.3) bifurcates from the \( \lambda \) axis. The eigenvalues of \( D_x f(0, 0) = A \) are \( \pm i \) and \( \pm ki \). Since \( A(\lambda) = D_x f(0, \lambda) = A + \lambda I \), these values lie on paths of eigenvalues, \( \lambda \pm i \) and \( \lambda \pm ki \), which cross the imaginary axis with nonzero speed at \( \lambda = 0 \). Thus, in this example, although we have two eigenvalues crossing the imaginary axis, only one family of orbits appears at \( \lambda = 0 \)—orbits with period \( 2\pi/k \). We argue, however, that the orbits on this family, \( \Gamma \), have virtual period \( 2\pi \).

Since the orbits on \( \Gamma \) are not isolated in \( \mathbb{R}^4 \times \{0\} \), we restrict our analysis to the unit sphere \( S^3 \) and the orbit \( \gamma \subset S^3 \). (The same analysis holds for each orbit on \( \Gamma \) if we restrict to the appropriate sphere \( S_r \), where \( r = ||x|| \).) Focusing on the Poincaré map \( \overline{T} \) associated with \( \gamma \) at \( x_0 \) on \( S^3 \), recall that \( \overline{T}^k \) has a fixed point \( x_0 \) of index \( k + 1 \). Now \( D_x \overline{T}^k(x_0) - I \) is nonsingular only if the zero \( x_0 \) of \( \overline{T}^k - \text{id} \) has index \( \pm 1 \). Since \( k \geq 2 \), \( D_x \overline{T}^k(x_0) - I \) is singular. Thus there exists a point \( y \in \mathbb{R}^2 \), \( y \neq 0 \), such that

\(^1\) Note that \((d/dt)(||x(t)||^2/2) - \langle x, \dot{x} \rangle - \langle x, F(x) \rangle + \lambda ||x||^2 \). Since \( F(x) \) is perpendicular to \( x \), \( ||x(t)|| \to 0 \) (as \( t \to \infty \)) for \( \lambda < 0 \), and \( ||x(t)|| \to \infty \) for \( \lambda > 0 \).
(D_x f^k(x_0) - I) y = 0. Equivalently, D_x f^k(x_0) y = y, and y has a virtual period of order k (i.e., the virtual period 2\pi). By applying the same argument to each S_r, every orbit on I is shown to have virtual period 2\pi.

**Example 2.** For \((u_1, u_2, v_1, v_2) \in \mathbb{R}^4\), let \(\dot{x} = Ax\) \((x \in \mathbb{R}^4)\) represent the linear system

\[
\begin{align*}
\dot{u}_1 &= u_2, \\
\dot{u}_2 &= -u_1, \\
\dot{v}_1 &= -kv_2, \\
\dot{v}_2 &= kv_1,
\end{align*}
\]

where \(k\) is a positive integer greater than 1. As in Example 1, the eigenvalues of \(A\) are \(\pm i\) and \(\pm ki\), and every point in \(\mathbb{R}^4\) lies on an orbit of (2.5). Orbits in the \(v\) plane have period \(2\pi/k\); all others have period \(2\pi\). Since this is a Hamiltonian system with total energy function \(H(u_1, u_2, v_1, v_2) = \frac{1}{2}(u_1^2 + u_2^2) - \frac{k}{2}(v_1^2 + v_2^2)\), the level sets (energy levels) of \(H\) are invariant under (2.5). For the level set \(H = c\), the orbits lie on tori (sometimes degenerate) of the form \(u_1^2 + u_2^2 = r\) and \(v_1^2 + v_2^2 = s\), where \(\frac{1}{2}(r - ks) = c\). Note that each level set is not compact. When \(c > 0\), the level set intersects the \(u\) plane (where \(s = 0\)) in a single orbit which has period \(2\pi\), and the set does not intersect the \(v\) plane. When \(c = 0\), the set intersects the \(u\) and \(v\) planes (where \(r = s = 0\)) only at the equilibrium 0. When \(c < 0\), the set intersects the \(v\) plane (where \(r = 0\)) in a single orbit which has period \(2\pi/k\), and the set does not intersect the \(u\) plane. For an orbit \(\gamma\) in the \(v\) plane, the associated Poincaré map \(T\) is periodic of order \(k\).

Again, we describe a perturbation of (2.5) which destroys all orbits except those of period \(2\pi/k\) in the \(v\) plane. First, identify orbits of (2.5), as in the first example. The projection of each level set is homeomorphic to \(\mathbb{R}^2\); the entire identification space is homeomorphic to \(\mathbb{R}^3\). For \((x_1, x_2, x_3) \in \mathbb{R}^3\), we let the plane \(x_3 = c\) be the projection of the set of points for which \(H = c\). Let \(\eta\) be a \(C^1\) vector field on \(\mathbb{R}^3\) such that \(dx_1/dt = dx_2/dt = 0\) for all \(x \in \mathbb{R}^3\); \(dx_2/dt = a\), where \(a \leq 0\), and \(dx_3/dt > 0\) for all other points. Hence, the images of the level sets are invariant, and the vector field is 0 only at the image of the short period orbits and at the origin. In the plane \(x_3 = c\), \(c \leq 0\), each zero \((0, 0, c)\) of \(\eta\) has index 0. Now, lift \(\eta\) to a \(C^1\) vector field \(g\) on \(\mathbb{R}^4\), perpendicular to the orbits of (2.5). The resulting system

\[
\dot{x} = F(x) = Ax + g(x)
\]

has orbits of (2.5) in the \(v\) plane and no other orbits. We can assume that \(g\) is \(O(\|x\|^2)\) at \(x = 0\), so that the eigenvalues of \(D_x F(0)\) are \(\pm i\) and \(\pm ki\). Note that \(H\) remains a first integral of (2.6) (i.e., the level sets of \(H\))
are invariant under (2.6). Let $T_k$ be the Poincaré map of an orbit $\gamma$ (in the $v$ plane) at the point $x_0$. We restrict $T_k$ to a 2-disk $D$ in the level set containing $\gamma$. (Again, $D$ is transverse to the $v$ plane.) Further analysis shows that the fixed point $x_0$ of $T_k$ has index $1 - k$. (See Fig. 2.)

Finally, we embed (2.6) in a parametrized system by setting

$$\dot{x} = f(x, \lambda) = F(x) + \lambda \text{ grad } H.$$ (2.7)

This system, like (2.4), has one family $\Gamma$ of orbits at $\lambda = 0$ and has no other orbits. Each orbit on $\Gamma$ has period $2\pi/k$. The eigenvalues $\pm i$ and $\pm ki$ of $D_xf(0, 0)$ lie on paths of eigenvalues, $\lambda \pm i$ and $-k\lambda \pm ki$, for the parametrized derivative $D_xf(0, \lambda) = A + (\text{grad } H) \lambda I$. Note that although both paths cross the imaginary axis with nonzero speed at $\lambda = 0$, $\lambda \pm i$ crosses from left to right, while $-k\lambda \pm ki$ crosses from right to left.

Again we restrict our analysis to the level set in $R^4$ containing $\gamma$. (The same argument holds for each orbit when we restrict to the appropriate level set.) Recall that the map $T_k$ associated with $\gamma$ has a fixed point $x_0$ of index $1 - k$. For $k \neq 2$, since $1 - k \neq \pm 1$, arguing as in the first example, we see that the orbits of $\Gamma$ also have virtual period $2\pi$. However, in the case $k = 2$, the fixed point $x_0$ has index $-1$ for $T_2$, and $D_xT^2(x_0) - I$ can be nonsingular. If it is, then $D_xT^2$ has no fixed points except 0, and $\gamma$ has no virtual periods other than the period $\pi$.

Throughout this theory, virtual periods of order 2 play a special role. In the next section we show that of the three possibilities mentioned at the beginning of this section for examples such as these, only (b1) and (b2) can occur—except when $k = 2$ and the paths of eigenvalues cross the imaginary axis (at $\pm i$ and $\pm 2i$) in opposite directions.

3. Definitions and Statement of the Main Theorem

In this section we examine the bifurcation of orbits from a path of stationary points of the general one-parameter system

$$\frac{dx}{dt} = f(x, \lambda),$$ (3.1)

Since $(d/dt)H(x(t)) = \langle \text{grad } H, F + \lambda \text{ grad } H \rangle - \lambda \|\text{grad } H\|^2$, we have $H(x(t))$ is strictly monotonic unless $x = 0$ or $\lambda = 0$. 

---

**Fig. 2.** The action of $T_k$ (as defined in Example 2) around the fixed point $x_0$ is shown for $k = 2$ and $k = 3$. 

$\begin{align*} &\text{k=2} \quad \text{k=3} \\
&\text{HOPF BIFURCATION} \quad 381 \\
&\text{FIG.} \quad 2. \text{ The action of } T_k \text{ (as defined in Example 2) around the fixed point } x_0 \text{ is shown for } k = 2 \text{ and } k = 3. 
\end{align*}$
where \( f \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n) \). Let \((x_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}\) be a zero of \( f \). If \( A = D_x f(x_0, \lambda_0) \) is nonsingular, then there is a path of zeroes \((x(\lambda), \lambda)\) of \( f \) through \((x_0, \lambda_0)\) for \( \lambda \) near \( \lambda_0 \). If, in addition, \( A \) has some purely imaginary eigenvalues, we say that \((x_0, \lambda_0)\) is a center of (3.1). For each pair \( \pm i \beta \) of imaginary eigenvalues of \( A \), the linear system

\[
\frac{dx}{dt} = Ax
\]  

(3.2)

has orbits of period \( 2\pi/\beta \). We call the (minimum) period of any (non-constant) orbit of (3.2) a center period of the center \((x_0, \lambda_0)\). Note that each center has only a finite number of center periods—the number being greater than or equal to the number of distinct purely imaginary eigenvalues of \( A \). For example, if \( n = 2 \) and \( A \) has eigenvalues \( \pm i/\beta \), then \( 2\pi/\beta \) is the only center period of \((x_0, \lambda_0)\). We say that a center \((x_0, \lambda_0)\) is isolated if there exists a neighborhood \( W \) of \((x_0, A)\) in \( \mathbb{R}^{n+1} \) such that \((x_0, A)\) is the only center of (3.1) in \( W \).

Let \( \{ \gamma_i \} \) be a sequence of orbits of (3.1), where \( \tau_i \) is the period of \( \gamma_i \), for each \( i \). If \( \lim_{i \to \infty} \gamma_i = (x_0, \lambda_0) \) and if the set \( \{ \tau_i \} \) is bounded, then we say there is a local bifurcation of orbits from \((x_0, \lambda_0)\). Let \( \tau = \lim_{i \to \infty} \tau_i \). When the set \( \{ \gamma_i \} \) lies on a component \( \Gamma \) of orbits of (3.1), we say that the family \( \Gamma \) bifurcates from \((x_0, \lambda_0)\) starting with period \( \tau \).

If there is a local bifurcation from \((x_0, \lambda_0)\), then \((x_0, \lambda_0)\) is a center and \( \tau \) is a center period of \((x_0, \lambda_0)\), (see [8, Appendix A]). In addition, suppose that \( \bar{\tau} \) is a virtual period of \( \gamma_i \), for each \( i \), and that \( \{ \bar{\tau}_i \} \) is bounded. Then a minor variation in the techniques of [8] show that \( \bar{\tau} = \lim_{i \to \infty} \bar{\tau}_i \) is also a center period of \((x_0, \lambda_0)\). In this case, we say that \( \Gamma \) bifurcates from \((x_0, \lambda_0)\) starting with virtual period \( \bar{\tau} \).

In the following, we assume that \((0, 0)\) is a center of (3.1). Since we are interested in local results here, we further assume that \( f(0, \lambda) = 0 \) for all \( \lambda \). Let \( \{ \pm i \beta_i \} \) be the set of distinct imaginary eigenvalues of \( A = D_x f(0, 0) \), where each \( i \beta_j \) lies on one or more paths of eigenvalues of \( A(\lambda) = D_x f(0, \lambda) \) (i.e., \( A = A(0) \)). In light of examples such as those in Section 2, we know that if \( \beta_j/\beta_j \) is an integer, there is not necessarily a bifurcation of orbits from \((0, 0)\) starting with period \( 2\pi/\beta_j \) (even when all other hypotheses of the Hopf Theorem are satisfied). In addition, if more than one path crosses the imaginary axis at the (necessarily multiple) eigenvalue \( i \beta_j \) when \( \lambda = 0 \), it is possible that no orbits bifurcate from the center (see Sect. 4).

The hypothesis that \( x_j(0) \neq 0 \) can be weakened (see [1]): the path \( x_j(\lambda) + i \beta_j(\lambda) \) must simply cross the imaginary axis (but not necessarily with nonzero speed). In our analysis, the direction of that crossing is also important. We formalize this idea in
DEFINITION 3.3. Let $E$ be the set of pure imaginary eigenvalues of $A$. For $i\beta \in E$, choose $\varepsilon > 0$ sufficiently small that the $\varepsilon$-ball about $i\beta$ in $C$, $B_\varepsilon(i\beta)$, contains no points of $E - \{i\beta\}$. Let $\mu(B_\varepsilon(i\beta), \lambda)$ be the number of eigenvalues (counting multiplicities) of $D_x(0, \lambda)$ in $B_\varepsilon(i\beta)$ with positive real part; and let

$$
\mu^+(i\beta) = \lim_{\lambda \to 0^+} \mu(B_\varepsilon(i\beta), \lambda)
$$

and

$$
\mu^-(i\beta) = \lim_{\lambda \to 0^-} \mu(B_\varepsilon(i\beta), \lambda).
$$

Then we define the crossing number of $i\beta$, $\chi(i\beta)$, as follows:

$$
\chi(i\beta) = \begin{cases} 
\mu^+(i\beta) - \mu^-(i\beta) & \text{if } i\beta \text{ is an eigenvalue of } A \\
0 & \text{otherwise.}
\end{cases}
$$

Loosely speaking, $\chi(i\beta)$ is the number of paths of eigenvalues (counted with multiplicities) that cross the imaginary axis in $C$ (at $i\beta$) from left to right minus the number that cross from right to left. Note that in Example 1 of Section 2, $\chi(i) = +1$ and $\chi(ki) = +1$, while in Example 2, $\chi(i) = +1$, and $\chi(ki) = -1$, i.e., $\chi(i) + \chi(ki) = 0$. It was shown in [1] that if $\chi(i\beta)$ is odd, for some eigenvalue $i\beta$ of $A$, then bifurcation occurs (without specifying the starting period). This result was strengthened in [4] to all cases where $\chi(i\beta) \neq 0$, again not specifying the minimum period of the bifurcating family.

The structure of families of orbits, their origin and/or termination in Hopf bifurcation, and their global continuation, is studied in [8] for a special class $G$ of functions in $C^1(R^n \times R, R^n)$. This class is described further in Section 5. For purposes of this discussion we mention that $G$ is dense in the weak $C^1$ topology (i.e., $C^1$ uniform convergence on each bounded set), and that a function $g \in G$ has the following properties:

1. Each center $(x_0, \lambda_0)$ of

$$
\dot{x} = g(x, \lambda)
$$

is simple; i.e., $D_x g(x_0, \lambda_0)$ has exactly one pair $\pm i\beta$ of purely imaginary eigenvalues, and these are algebraically simple. In addition, $i\beta$ lies on a path of eigenvalues $\alpha(\lambda) + i\beta(\lambda)$ of $A(\lambda)$ such that $\alpha'(\lambda_0) \neq 0$. We call such a center a simple Hopf point. For each simple Hopf point $(x_0, \lambda_0)$ there exists (by the Hopf Theorem) a smooth path of orbits of (3.4) which bifurcates from $(0,0)$ starting with period $2\pi/\beta$, and along which the periods vary continuously.
Either $\Gamma$ is unbounded in $(x, \lambda)$ space; the periods of orbits on $\Gamma$ are unbounded; or $\Gamma$ terminates in one of two ways:

(i) $\Gamma$ ends at another simple Hopf point of (3.5), or

(ii) $\Gamma$ ends at a bifurcation orbit $\rho$ at which the period suddenly drops by a factor of two. That is, for a sequence $\{y_i\}_{i \in \mathbb{N}}$ of orbits on $\Gamma$ such that $\lim_{i \to \infty} y_i = \rho$, let $\tau_i$ be the period of $y_i$, for each $i$, and let $\tau = \lim_{i \to \infty} \tau_i$. Then the period of $\rho$ is $\frac{1}{2}\tau$. Along with $\Gamma$, two other paths of orbits emanate from $\rho$—both having orbits of period approximately $\frac{1}{2}\tau$ near $\rho$. This type of orbit bifurcation is commonly called a period-doubling bifurcation (or, in our case, following along $\Gamma$, a period-halving bifurcation).

In the following, we assume that $(0, 0)$ is an isolated center of the general system (3.1) (in particular, $A = D_x f(0, 0)$ is nonsingular); and we assume that $f(0, 0) \equiv 0$. To investigate the local bifurcation of orbits of (3.1) from $(0, 0)$, we consider a sequence $\{g_i\}_{i \in \mathbb{N}}$ of functions in $G$ such that $\lim_{i \to \infty} g_i = f$ and analyze how bifurcating families of orbits of $1 = g_i(x, \lambda)$ (3.5) converge as $i \to \infty$. Let $\{i \beta_j\}_{1 \leq j \leq m}$ be the set of imaginary eigenvalues of $A$. (We repeat values $r$ times in this set for eigenvalues of multiplicity $r$.) For each function $g_i$, as above, the center $(0, 0)$ of (3.1) splits into $m$ simple Hopf points of (3.5). We assume that these Hopf points lie on the $\lambda$ axis at values $\{(0, \lambda_{ij})\}_{1 \leq j \leq m}$, for each $i$. Thus $\lim_{i \to \infty} \{(0, \lambda_{ij})\} = (0, 0)$. Let $\Gamma_{ij}$ be the path of orbits of (3.5) such that $\Gamma_{ij}$ bifurcates from $(0, \lambda_{ij})$ starting with period $2\pi/\beta_j$.

Focusing on one eigenvalue $i \beta_j$ of $A$ and the associated sequence $\{\Gamma_{ij}\}_{i \in \mathbb{N}}$, let $\gamma_{ij}$ be an orbit of (3.5) on $\Gamma_{ij}$, and let $\tau_{ij}$ be the period of $\gamma_{ij}$, for each $i$. Consider those sequences $\{\gamma_{ij}\}_{i \in \mathbb{N}}$ for which the associated set $\{\tau_{ij}\}$ of periods is bounded, and let

$$Q_r = \{x: x = \lim_{i \to \infty} x_{ir}, \text{ where } x_{ir} \in \gamma_{ij}, \text{ for each } i\}.$$

There are essentially three possibilities for $Q_r$:

(1) $Q_r$ is a set of orbits of (3.1), and $Q_r$ bifurcates from $(0, 0)$ starting with period $2\pi/\beta_r$.

(2) $Q_r$ is a set of orbits of (3.1), and $Q_r$ bifurcates from $(0, 0)$ starting with virtual period $2\pi/\beta$, and with (minimum) period $2\pi/k \beta_r$, for some integer $k > 1$.

(3) $Q_r$ is empty.

In Section 5 we prove that if $Q_r$ is nonempty, then (1) or (2) must hold.
(Note that, since the period of an orbit is also a virtual period, case (1) can be included under (2) by allowing $k = 1$.) Here we motivate the statement and proof of the main theorem (Theorem 3.6), which gives sufficient conditions for $Q_r$ to be nonempty. For each orbit $\psi_r$ sufficiently near $(0,0)$ on $Q_r$, there is a sequence $\{\gamma_{ir}\}_{i \in N}$ of orbits where $\gamma_{ir}$ is on $\Gamma_{ir}$, for each $i$, and $\lim_{i \to \infty} \gamma_{ir} = \psi_r$. Let $\tau_{ir}$ be the period of $\gamma_{ir}$ for each $i$, and let $\tau_r$ be the period of $\psi_r$. Then $\lim_{i \to \infty} \tau_{ir} = k\tau_r$, for some integer $k \geq 1$. It was shown in [5] that $k\tau_r$ is a virtual period of the limit orbit $\psi_r$. Thus $Q_r$ bifurcates from $(0,0)$ starting with virtual period $2\pi/k\beta_r$ and with (minimum) period $2\pi/k\beta_r$. In this case, there is an eigenvalue $i\beta$, of $A$, such that $\beta$ is an integer multiple of $k\beta_r$.

If $Q_r$ is empty, then (for $i$ sufficiently large) $\Gamma_{ir}$ terminates either at another Hopf point or in a period-doubling bifurcation orbit. Suppose that $\Gamma_{ir}$ terminates at another Hopf point $(0, \lambda_{ir})$, so that $\Gamma_{ir} = \Gamma_{is}$. As $i \to \infty$, the sequences $\{(0, \lambda_{ir})\}$ and $\{(0, \lambda_{is})\}$ converge to $(0,0)$. If all orbits on $\Gamma_{is} = \Gamma_{is}$ also converge to $(0,0)$, then necessarily $\beta = \beta_s$. (For an example of this phenomenon for Hamiltonian systems, see D. Schmidt [11]. This example is mentioned again in Sect. 4.) For simple Hopf points, all crossing numbers are $\pm 1$. It was shown in [8] that if $\Gamma_{ir} = \Gamma_{is}$, then $\chi(i\beta_r) + \chi(i\beta_s) = 0$. Thus if $\chi(i\beta_r) \neq 0$ (in the general system (3.1)), then some generic family $\Gamma$ associated with the eigenvalue $i\beta_r$ does not terminate in another center also associated with $i\beta_s$.

The second situation in which $Q_r$ might be empty occurs when $\Gamma_{ir}$ terminates at a bifurcation orbit. Suppose, for example, that $i\beta_s = 2i\beta_r$ for some eigenvalue $i\beta_s$ of $A$; thus $\pi/\beta_r = 2\pi/\beta_s$ is a center period of $(0,0)$. ($\Gamma_{is}$ bifurcates from $(0, \lambda_{is})$ starting with period $\pi/\beta_r$.) Suppose further that $\Gamma_{ir}$ and $\Gamma_{is}$ meet in a period-doubling bifurcation at orbit $\rho_i$, for each $i$. As $i \to \infty$, the sequence $\{\rho_i\}_{i \in N}$ may converge to $(0,0)$, along with $\{\Gamma_{ir}\}_{i \in N}$. Note, however, that $\Gamma_{is}$ continues through the third branch of orbits emanating from $\rho_i$, for each $i$; hence $Q_r = \lim_{i \to \infty} \Gamma_{is}$ bifurcates from $(0,0)$ starting with period $\pi/\beta_r$. Example 2 of Section 2 illustrates this case.

Details of the previous arguments appear in Section 5.

**THEOREM 3.6.** Let $(0,0)$ be an isolated center of (3.1), and let $i\beta$ be an eigenvalue of $A = D_x f(0,0)$. If $\chi(i\beta) \neq 0$, and if $\pi/\beta$ is not a center period of $(0,0)$, then there exists a family $Q$ of orbits of (3.1) such that $Q$ bifurcates from $(0,0)$ starting with virtual period $2\pi/\beta$.

**Remark.** We use the hypothesis that $\pi/\beta$ not be a center period of $(0,0)$ rather than any hypothesis on the eigenvalue $2i\beta$, since $\pi/\beta$ can be center period even when $2i\beta$ is not an eigenvalue of $A$.

**COROLLARY.** If $A$ has only two pairs of purely imaginary eigenvalues $\pm i\beta_r$ and $\pm i\beta_s$ such that $\beta_s/\beta_r$ is an integer, if $\chi(i\beta_r) \neq 0$, and if $\chi(i\beta_s) + \chi(i\beta_r)$
\(x(2i\beta_r) \neq 0,\) then there exists a family \(Q\) of orbits of (3.1) such that \(Q\) bifurcates from \((0, 0)\) starting with virtual period \(2\pi/\beta_r.\)

Proof of Corollary. If only one eigenvalue is an integer multiple of \(i\beta_r,\) then a necessary condition for \(\pi/\beta\) to be a center period of \((0, 0)\) is that \(2i\beta\) be an eigenvalue of \(A.\) As in the previous argument, the only way that \(Q\) can fail to have orbits of virtual period \(2\pi/\beta,\) is if the nearby families \(T_{ir}\) and \(T_{iis}\) meet in a period-doubling bifurcation. A necessary condition for this bifurcation to occur is that \(\chi(i\beta_r) + \chi(2i\beta_r) = 0.\)

4. A Center Theorem for Hamiltonian Systems

In this section we examine the bifurcation of orbits from isolated centers of Hamiltonian systems. For \((x, y) \in \mathbb{R}^{2n}\) and a \(C^2\) function \(H: \mathbb{R}^{2n} \to \mathbb{R},\) Hamilton's equations are

\[
\dot{x} = H_y, \quad \dot{y} = -H_x. \tag{4.1}
\]

For \(z \in \mathbb{R}^{2n}\) we express this system as

\[
\dot{z} = f(z) = JH_z, \tag{4.1}
\]

where \(J\) is the \(2n \times 2n\) matrix \((0, I)\) and \(H_z\) is the gradient of \(H.\) In the following, we assume that \(z = 0\) is an isolated center of (4.1), in particular, \(A = Df(0) = JH_z(0)\) is nonsingular and has at least one pair of pure imaginary eigenvalues \(\pm i\beta.\) We begin with the classical Liapunov Center Theorem (see, e.g., [12]):

**Theorem (L.C.T.).** Suppose that \(\pm i\beta\) are simple eigenvalues of \(A,\) and that if \(\mu\) is another eigenvalue of \(A,\) then \(\mu / i\beta \neq \text{integer}.\) Then a one parameter family of orbits of (4.1) bifurcates from \(z = 0\) starting with period \(2\pi/\beta.\)

In the case of resonance (where some eigenvalue \(\mu\) is an integer multiple of \(i\beta)),\) the conclusion of this theorem may not hold. An example of this phenomenon (for Hamiltonians) appears in [10, pp. 109–110]. Here \(\pm i\) and \(\pm 2i\) are eigenvalues of \(A,\) but only one family of orbits bifurcates from \(z = 0\) (starting with period \(\pi).\) Another possibility is illustrated in an example of D. Schmidt [11]. In this example, the only purely imaginary eigenvalues of \(A\) are \(\pm i;\) however, these eigenvalues are not simple (each has multiplicity 2), and no orbits bifurcate from \(z = 0.\) Later in this section we describe a method of embedding Hamiltonians into parametrized systems.
In such an augmentation of the system in Schmidt's example, we find (by Theorem 3.6) that \( \chi(i) = 0 \).

Some recent papers on Hamiltonian systems have specifically addressed the resonance problem. In the following, we use the term “period-multiple” to refer to any nonzero integer multiple of the period of an orbit. A. Weinstein [13] eliminated the condition on the imaginary eigenvalues of \( A \) by assuming that the Hessian \( H_{zz}(0) \) is positive definite. In this case, he showed that each energy level \( H = \varepsilon \), for small \( \varepsilon > 0 \), possesses at least \( n \) distinct orbits of (4.1)—each orbit with some period-multiple near that of an orbit of \( \dot{z} = AZ \). J. Moser [9] generalized Weinstein's result by focusing on linear subspaces of \( R^{2n} \) for which all orbits of \( \dot{z} = AZ \) have period-multiple \( T \), for some \( T > 0 \). In particular, let \( R^{2n} = E_1 \oplus E_2 \), where \( E_1 \) and \( E_2 \) are invariant under \( \dot{z} = AZ \), and all orbits in \( E_1 \) have period-multiple \( T \), while no orbits in \( E_2 \) have period-multiple \( T \). Moser showed that if \( H_{zz}(0) \) is positive definite on \( E_1 \), then each energy level \( H = \varepsilon \), for small \( \varepsilon > 0 \), has at least \( \frac{1}{2} \dim E_1 \) distinct orbits of (4.1) of period multiple near \( T \). Along these lines, E. Fadell and P. Rabinowitz [6] proved that if the signature \( 2\nu \) of \( H_{zz}(0) \) restricted to \( E_1 \) is nonzero, then either:

(i) there are orbits of (4.1) of period-multiple \( T \) in every neighborhood of \( z = 0 \).

(ii) there exists a pair of integers \( k, m \geq 0 \) with \( k + m \geq |\nu| \), and a left neighborhood \( I_L \), and a right neighborhood \( I_R \), of \( T \) in \( R \) such that for all \( t \in I_L \) (resp. \( I_R \)), (4.1) has at least \( k \) (resp. \( m \)) distinct orbits of period-multiple \( t \).

Again, it should be noted that in each of these results no mention is made of the minimum periods of orbits, or of the orbits lying on connected families. The techniques used do not distinguish periods and period-multiples.

For us to be able to use the theory of parametrized systems here, we embed the Hamiltonian system (4.1) into a parametrized system at \( \lambda = 0 \). (This technique first appeared in [1].) For \( z \in R^{2n} \) and \( \lambda \in R \), let

\[
\dot{z} = F(z, \lambda) = (J + \lambda) H_z.
\]  

(4.2)

Note that \( F(z, 0) = f(z) \) and that \( F(0, \lambda) = 0 \), for all \( \lambda \). Furthermore, since

\[
\frac{d}{dt} H(x(t), y(t)) = H_x \frac{dx}{dt} + H_y \frac{dy}{dt} = \lambda (H_x^2 + H_y^2),
\]

for \( \lambda > 0 \) (resp. \( \lambda < 0 \)), \( H \) is strictly increasing (resp. decreasing) along solution curves of (4.2)—unless \( H_x^2 + H_y^2 = 0 \) (i.e., unless \( (x, y) \) is a stationary point). Thus nonconstant periodic orbits of (4.2) can occur only
at $\lambda = 0$. In addition, suppose that $D_2F(0, \lambda)$ has purely imaginary eigenvalues at $\lambda = \lambda_0$. Consider the linear Hamiltonian system $\dot{z} = Az = D_2F(0, 0)$ and embed this system into $\dot{z} = (J + \lambda)H_2(0)(z) = D_2F(0, \lambda)$. Then the parametrized system has orbits at $\lambda = \lambda_0$. Hence $\lambda_0 = 0$. We re-state these facts as propositions for later reference:

**Proposition 4.3** [1]. (i) Orbits of (4.2) can occur only at $\lambda = 0$.

(ii) In (4.2) purely imaginary eigenvalues of $D_2F(0, \lambda)$ can occur only at $\lambda = 0$.

Starting with the hypotheses of the L.C.T. and embedding the Hamiltonian system into (4.2) as above, D. Schmidt [12] verified the hypotheses of the Hopf Theorem for (4.2) and obtained the L.C.T. as a consequence. Similarly, we obtain the following Center Theorem from Theorem 3.6:

**Theorem 4.4.** Suppose that for the Hamiltonian system (4.1), $A = D_2f(0)$ is nonsingular and has a pair of purely imaginary eigenvalues $\pm i\beta$ (of arbitrary multiplicity). Let $E_\beta \subset \mathbb{R}^{2n}$ be the eigenspace corresponding to $\pm i\beta$. If the signature $\sigma$ of the quadratic form $(H_{2z}(0)v, v)$, $v \in E$, is non-zero, and if $\pi/\beta$ is not a center period of 0, then a family $\Gamma$ of orbits of (4.1) bifurcates from $z = 0$ starting with virtual period $2\pi/\beta$.

**Proof.** We embed (4.1) into the parametrized system (4.2) and relate the signature $\sigma$ to the crossing number $\chi(i\beta)$ in (4.2). Namely, $\sigma = 2\chi(i\beta)$. Since $\sigma \neq 0$, $\chi(i\beta) \neq 0$. Theorem 3.6 guarantees the existence of a family $\Gamma$ of orbits of (4.2) which bifurcates from $(0,0)$ starting with virtual period $2\pi/\beta$. By Proposition 4.3(i), nonconstant orbits of (4.2) can only occur at $\lambda = 0$; hence $\Gamma$ is the desired family of orbits of (4.1).  

We remark again that the set of virtual periods of an orbit is a special, finite subset of the set of all period-multiples.

5. **Proof of the Main Theorem**

In this section we prove Theorem 3.6, after some preliminary lemmas. First, we describe the class of functions in $C^3(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ which are used in the limit arguments of the proof, as outlined in Section 3. For a point $(x_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ on an orbit $\gamma$ of (3.1), let $T$ be the Poincaré return map, defined on an $n-1$ disk in $\mathbb{R}^n \times \{\lambda_0\}$ transverse to $\gamma$ at $(x_0, \lambda_0)$. A multiplier of $\gamma$ is an eigenvalue of $D_xT(x_0, \lambda_0)$. Let $K$ be the subset of $C^3(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ such that $g \in K$ if every orbit of

$$\dot{x} = g(x, \lambda)$$

(5.1)
is one of the following three types:

(0) orbits having no multipliers which are roots of unity;

(I) bifurcation orbits having a simple multiplier equal to +1 and no other multipliers that are roots of unity, and from which precisely two arcs of orbits emanate; or

(II) bifurcation orbits having a simple multiplier equal to −1 and no other multipliers that are roots of unity, and from which three arcs of orbits emanate—one arc with orbits whose periods are approximately twice as long as those on the other two arcs.

Each orbit of type (0), (I), or (II) lies on a path of orbits of (5.1) along which the periods vary continuously. (The two low-period arcs emanating from a type (II) orbit form such a path). A demonstration that \( K \) is residual in \( C^3(R^n \times R^m) \) is given in [2, Appendix A] and is based on earlier results of Brunovský and Peixoto.

Since we are interested in local results here, we use the weak \( C' \)-topology for the proof of Theorem 3.6 and focus on a compact neighborhood \( B \) of \( (0, 0) \) in \( R^n \times R \); i.e., \( \{g_n\} \to f \) on \( B \) means \( \{g_n(x)\} \) along with the first \( r \) derivatives converges uniformly to \( f(x) \) and its derivatives for \( x \in B \). In addition, when we refer to an orbit \( \gamma \) of (3.1) in \( B \), we mean that \( \gamma \), as a subset of \( R^n \times R \), is (totally) contained in \( B \).

Finally, let \( G \) be the subset of \( C^3(R^n \times R, R^n) \) such that \( g \in G \) if the following properties hold:

(P1) every orbit of (5.1) in \( B \) is of type (0), (I), or (II); and

(P2) each center of (5.1) in \( B \) is a simple Hopf point, (for definition, see Sect. 3).

**Lemma 5.2.** \( G \) is residual in \( C^3(R^n \times R, R^n) \).

**Proof.** The set \( K \) of functions that satisfy property (P1) is residual in \( C^3(R^n \times R, R^n) \). The set \( L \) of functions that satisfy property (P2) is open and dense in \( C^3(R^n \times R, R^n) \). Hence, \( G = K \cap L \) is residual in \( C^3(R^n \times R, R^n) \).

**Corollary.** \( G \) is dense in \( C^1(R^n \times R, R^n) \).

Consider again the general system (3.1), under the assumptions imposed on \( f \) in Section 3. In particular, for a neighborhood \( W \) of \( (0, 0) \) in \( R^{n+1} \), we assume that \( (0, 0) \) is the only center of (3.1) in \( W \). For \( g \in G \) sufficiently near \( f \), the center \( (0, 0) \) splits into \( m \) (where \( m \leq n \)) simple Hopf points of (5.1), each in \( W \). The eigenvalues of \( D_x g \) at these centers are near those of \( A = D_x f(0, 0) \). More explicitly, let \( \{g_j\}_{j \in N} \) be a sequence of maps in \( G \)
such that \( \lim_{j \to \infty} g_j = f \) on \( \overline{W} \). We assume that the zeroes of each \( g_j \) are identical to those of \( f \) in \( W \). (This condition on \( g_j \) is obtained in the proof of Theorem 3.6.) Then for \( j \) sufficiently large,

\[
\dot{x} = g_j(x, \lambda)
\]  

(5.3)

has exactly \( m \) centers \( \{(0, \lambda_{jk})\}_{1 \leq k \leq m} \) in \( W \) such that \( \lim_{j \to \infty} (0, \lambda_{jk}) = (0, 0) \), for each \( k \), \( 1 \leq k \leq m \) (i.e., there exists a bound \( N > 0 \) such that the value of \( m \) is constant for \( j > N \)). Fix \( j \), and let \( D_j = \{(0, \lambda_{jk})\}_{1 \leq k \leq m} \). For each \( k \), \( D_x g_j(0, \lambda_{jk}) \) has exactly one pair \( \pm i \beta_{jk} \) of purely imaginary eigenvalues. For some \( d \leq m \), let \( E_j = \{i \beta_{jk}\}_{1 \leq k \leq d} \) be the set of these eigenvalues such that \( \lim_{j \to \infty} i \beta_{jk} = i \beta \), for each \( k \), \( 1 \leq k \leq d \). (Again, \( d \) is constant for \( j \) sufficiently large.)

Recall that \( \chi(i \beta) \) is defined to be the number of paths of eigenvalues of \( D_x f(0, \lambda) \) (near \( \lambda = 0 \)) that cross the imaginary axis \( I \) near \( i \beta \) from left to right (in the complex plane) minus the number that cross from right to left. For small perturbations of \( f \), the corresponding paths still cross \( I \) at values near \( i \beta \)—and these crossings occur in the same directions. Thus we have the following invariance property:

**Lemma 5.4.** For \( j \) sufficiently large,

\[
\chi(i \beta) = \sum_{k=1}^{d} \chi(i \beta_{jk}),
\]

summing over \( i \beta_{jk} \in E_j \).

(Note that (3.1) is used in calculating \( \chi(i \beta) \); and (5.3), in calculating each \( \chi(i \beta_{jk}) \).

We restate the following facts (from Sect. 3) in a lemma for future reference:

**Lemma 5.5 ([5] and [8]).** For a sequence \( \{\gamma_i\}_{i \in N} \) of orbits of (5.3), let \( \tau_i \) be a virtual period of \( \gamma_i \), for each \( i \). Assume that \( \{\gamma_i\}_{i \in N} \) converges to an orbit \( \rho \) or to a stationary point \((x_0, \lambda_0)\) of (3.1), as \( i \to \infty \). Assume further that \( \{\tau_i\}_{i \in N} \) is bounded so that some subsequence converges to \( \tau \). If \( \{\gamma_i\} \to \rho \), then \( \tau \) is a virtual period of \( \rho \); if \( \{\gamma_i\} \to (x_0, \lambda_0) \) and \( D_x f(x_0, \lambda_0) \) is non-singular (i.e., \( x_0, \lambda_0 \) is a center), then \( \tau \) is a center period of \( (x_0, \lambda_0) \).

**Proof of Theorem 3.6.** Section 3 contains motivational ideas for this proof. We follow the same notation, when possible. In addition, for a sequence \( \{P_i\}_{i \in N} \) of sets, we use the notation \( \lim_{i \to \infty} P_i \) to mean the set of limit points of sequence \( \{x_i\}_{i \in N} \), where \( x_i \in P_i \), for each \( i \).

For some positive integer \( q \), let \( \{v_i\}_{1 \leq i \leq q} \) be the set of distinct center
periods of \((0,0)\). We assume that \(v_1 = 2\pi/\beta\). Let \(p = \max_{1 \leq i \leq q}\{v_i\}\); and let \(\delta = \min\{\rho_1, \rho_2, 1\}\), where

\[
\rho_1 = \begin{cases} 
\min_{2 \leq i \leq q} \{|v_1 - v_i|\} & \text{if } q > 1 \\
1 & \text{if } q = 1
\end{cases}
\]

and

\[
\rho_2 = \min_{1 \leq i \leq q} \{|\frac{1}{2}v_1 - v_i|\}.
\]

Since \(\frac{1}{2}v_1 = \pi/\beta\) is not a center period, both \(\rho_1\) and \(\rho_2\) are strictly positive. Also note that \(\delta < p\). Recall that we assume \(f(0, \lambda) \equiv 0\). Since \(A\) is non-singular, there exists a neighborhood \(U \subset W\) of \((0,0)\) in \(R^{n+1}\) such that \(D_x f(0, \lambda)\) is nonsingular for \((0, \lambda) \in U\).

Let \(B_\varepsilon\) be the \(\varepsilon\)-ball about \((0,0)\) in \(R^{n+1}\). Choose \(\varepsilon\) sufficiently small so that

1. \(B_{2\varepsilon} \subset U\),
2. if \(y\) is an orbit of (3.1) in \(\overline{B_\varepsilon}\), and \(\tau\) is a virtual period of \(y\), then either \(\tau > 3p\) or \(|\tau - v_i| < \frac{1}{2}\delta\), for some center period \(v_i\) of \((0,0)\).

To show that \(\varepsilon\) can be chosen to satisfy condition (2), we argue as follows. Assume no such \(\varepsilon\) exists. Then there is a decreasing sequence \(\{\varepsilon_j\}_{j \in N}\) of positive numbers and a sequence \(\{\gamma_j\}_{j \in N}\) of orbits of (3.1) such that \(\gamma_j \subset \overline{B_\varepsilon}\), for each \(j\). Thus \(\lim_{j \to \infty} \gamma_j = (0, 0)\). Let \(\tau_j\) be a virtual period of \(\gamma_j\), for each \(j\), where \(\tau_j \leq 3p\) and \(|\tau_j - v_i| \geq \delta\), for each center period \(v_i\). By Lemma 5.5, \(\tau = \lim \tau_j\) is a center period of \((0,0)\). But \(|\tau - v_i| \geq \delta\), \(1 \leq i \leq q\), a contradiction.

Following the choice of \(\varepsilon\), let \(\{g_j\}_{j \in N}\) be a sequence of maps in \(G\) such that \(\lim_{j \to \infty} g_j = f\) on \(\overline{B_\varepsilon}\), and \(\|f - g_j\|/\|f\| < 1\) on \(\overline{B_\varepsilon}\), for each \(j\). Since \(\|f - g_j\|/\|f\| < 1\), the zeroes of each \(g_j\) are identical with the zeroes of \(f\) in \(B_\varepsilon\). Thus for \((x, \lambda) \in B_\varepsilon\), \(g_j(x, \lambda) = 0\) implies \(x = 0\). Let \(D_j\) and \(E_j\) be defined as for Lemma 5.4. We choose \(j\) sufficiently large that the following conditions hold:

1. \(D_j \subset B_\varepsilon\),
2. \(D_x g_j(0, \lambda)\) is nonsingular, for \((0, \lambda) \in B_\varepsilon\),
3. If \(\gamma_j\) is an orbit of (5.3) in \(\overline{B_\varepsilon}\) and \(\tau_j\) is the period of \(\gamma_j\), then \(\tau_j > 3p\) or \(|\tau_j - v_i| < \frac{1}{4}\delta\), for \(1 \leq i \leq q\).

We argue that a bound \(N > 0\) can be chosen so that condition (3) is satisfied by \(j > N\). Assume no such \(N\) exists. Then there exists a sequence \(\{g_j\}\) of maps in \(G\) such that \(\lim_{j \to \infty} g_j = f\) on \(\overline{B_\varepsilon}\), and such that for each \(j\), \(\gamma_j\) is an orbit of (5.3) in \(\overline{B_\varepsilon}\) with period \(\tau_j\), where \(\tau_j \leq 3p\) and \(|\tau_j - v_i| \geq \frac{1}{4}\delta\),
As $j \to \infty$, $\{\gamma_j\}_{j \in \mathbb{N}}$ converges to (i) an orbit $\gamma$ of (3.1) in $B_\epsilon$ or (ii) a stationary point (necessarily the center $(0, 0)$) of (3.1) in $B_\epsilon$. By Lemma 5.5, in case (i), $\tau = \lim_{j \to \infty} \tau_j$ is a virtual period of $\gamma$; in case (ii), $\tau$ is a center period of $(0, 0)$. In either case, since $\tau \leq 3p$ and $|\tau - v_i| > \frac{1}{4}\delta$, for $1 \leq i \leq q$, we have a contradiction.

Fix $j$. By the Hopf Theorem, a path $\Gamma_{jr}$ of orbits bifurcates from each center $(0, \lambda_{jk})$, $1 \leq k \leq m$, of (5.3) starting with period $2\pi/\beta_{jk}$ and along which the periods vary continuously. Note that condition (3) implies

$$\text{(3') for each center } (0, \lambda_{jk}) \in D_j, \text{ the center period } 2\pi/\beta_{jk} \text{ satisfies } |(2\pi/\beta_{jk}) - v_i| \leq \frac{1}{4}\delta, \text{ for some } i, 1 \leq i \leq q.$$ 

Again, let $E_j = \{i\beta_{jr}\}_{1 \leq r < \ell}$ be the set of eigenvalues of $D_x g_j(0, \lambda_{jr})$, for each $j$, such that $\lim_{j \to \infty} i\beta_{jr} = i\beta$, for each $r$. We claim that if $\gamma$ is an orbit of (5.3) with period $\tau$, and $\gamma \in \Gamma_{jr} \cap \overline{B}_\epsilon$, then $\tau \in [(2\pi/\beta) - \frac{1}{2}\delta, (2\pi/\beta) + \frac{1}{2}\delta]$. The claim follows these observations:

(i) $\tau < 3p$ or $|\tau - v_i| < \frac{1}{4}\delta$, for some $i$, $1 \leq i \leq q$ (by condition (3) on choice of $j$);

(ii) there are orbits on $\Gamma_{jr} \cap B_\epsilon$ with periods arbitrarily close to $2\pi/\beta_{jk}$;

(iii) $|(2\pi/\beta_{jr}) - v_i| \leq \frac{1}{4}\delta$, for some $i$, $1 \leq i \leq q$ (by condition (3')); and

(iv) $|(2\pi/\beta) - v_i| > \delta$, for each $v_i$, $2 \leq i \leq q$ (by the definition of $\delta$).

As $j \to \infty$, $\{i\beta_{jr}\} \to i\beta$, which implies $\{2\pi/\beta_{jr}\} \to 2\pi/\beta$. Thus, by (iv), the center period $v_i$ in (iii) must be $2\pi/\beta = v_1$. By (i) $|\tau - v_1| < \frac{1}{4}\delta$.

It was shown in [8] that there are four possibilities for each $\Gamma_{jr}$:

1. $\Gamma_{jr}$ is unbounded in $(x, \lambda)$-space;
2. the periods of orbits on $\Gamma_{jr}$ are unbounded;
3. $\Gamma_{jr}$ ends in a type (II) bifurcation orbit; or
4. $\Gamma_{jr}$ ends at another simple Hopf point of (5.3).

In case (1), $\Gamma_{jr}$ extends to the boundary of $B_\epsilon$ (i.e., $\Gamma_{jr}$ intersects $\partial B_\epsilon$). In case (2), by the bounds on periods of orbits on $\Gamma_{jr} \cap \overline{B}_\epsilon$, $\Gamma_{jr}$ must also extend to $\partial B_\epsilon$. In case (3), $\Gamma_{jr}$ ends in a type (II) orbit $\rho$. If $\rho \in \overline{B}_\epsilon$, then $\rho$ has period $\tilde{\tau}$ in $[(\pi/\beta) - \frac{1}{2}\delta, (\pi/\beta) + \frac{1}{2}\delta]$; i.e., $|\tilde{\tau} - \frac{1}{2}v_i| \leq \frac{1}{2}\delta$. But $|\tilde{\tau} - v_i| \leq \frac{1}{4}\delta$, for some $v_i$ ($1 \leq i \leq q$), and $|v_i - \frac{1}{2}v_i| > \delta$ implies $\frac{1}{2}\delta \geq |v_i - \frac{1}{2}v_i| - |\tilde{\tau} - \frac{1}{2}v_i| > \delta - \frac{1}{8}\delta = \frac{7}{8}\delta$, a contradiction. Thus $\rho \notin \overline{B}_\epsilon$, and again $\Gamma_{jr}$ extends to $\partial B_\epsilon$.

In case (4), either $\Gamma_{jr}$ extends to $\partial B_\epsilon$ or $\Gamma_{jr} \subset B_\epsilon$, and $\Gamma_{jr}$ terminates at another center $(0, \lambda_{js})$ ($1 \leq s \leq m$, $s \neq r$) of (5.3); i.e., $\Gamma_{jr} = \Gamma_{js}$ bifurcates from $(0, \lambda_{js})$, starting with period $2\pi/\beta_{js}$. By the bounds on the periods of orbits in $\Gamma_{jr} \cap B_\epsilon$, we see that since $|(2\pi/\beta_{js}) - v_i| \leq \frac{1}{4}\delta$ for some $v_i$ ($1 \leq i \leq q$), then $v_i = v_1$ and $i\beta_{js} \in E_j$. It was shown in [8] (for $g_j \in G$) that if
the two simple Hopf points \((0, \lambda_{jr})\) and \((0, \lambda_{jr'})\) \((r \neq s)\) are connected by a path of orbits, as in this case by \(\Gamma_{jr} = \Gamma_{js},\) then \(\chi(i\beta_{jr}) + \chi(i\beta_{js}) = 0.\) If each \(\Gamma_{jr}\) (corresponding to \(i\beta_{jr} \in E_j\)) is contained in \(B_e\), by counting pairs, we have \(\sum_{r=1}^{d} \chi(i\beta_{jr}) = 0.\) However, by Lemma 5.4, \(\chi(i\beta) = \sum_{r=1}^{d} \chi(i\beta_{jr})\) (summing over all \(i\beta_{jr} \in E_j\)). Since \(\chi(i\beta) \neq 0,\) it is necessarily the case that for some \(i\beta_{jr} \in E_j,\) \(\Gamma_{jr}\) bifurcates from a center \((0, \lambda_{jr})\) starting with period \(2\pi/\beta_{jr}\), and \(\Gamma_{jr}\) extends to \(\partial B_e.\)

Thus we have shown in each of cases (1)–(4) (and for each \(j\)) the existence of a path \(\Gamma_{jr}\) which bifurcates from center \((0, \lambda_{jr})\) starting with period \(2\pi/\beta_{jr}\), where \(i\beta_{jr} \in E_j,\) and which extends to \(\partial B_e.\) As \(j \to \infty,\) \(\{(0, \lambda_{jr})\} \to (0, 0)\) and \(\{i\beta_{jr}\} \to i\beta.\) Let \(\Gamma_{jr} = \Gamma_{jr} \cap B_e,\) and let \(\{\gamma_j\}_{j \in N}\) be a sequence of orbits of (5.3) such that \(\gamma_j \subset \Gamma_{jr}\) and \(\gamma_j\) has period \(\tau_{jr}\), for each \(j.\) Since there is a bound on the periods of all orbits on \(\Gamma_{jr},\) \(\{\tau_{jr}\}_{j \in N}\) is bounded. By Lemma 5.5, \(\lim_{j \to \infty} \tau_{jr}\) is either an orbit \(\rho\) of (3.1) with virtual period \(\tau = \lim_{j \to \infty} \tau_{jr}\) or a stationary point \((x_0, \lambda_0)\) of (3.1). If it is a stationary point, then \((x_0, \lambda_0)\) is a center of (3.1) in \(B_e\) and hence must be \((0, 0).\) However, all orbits in \(\{\Gamma_{jr}\}_{j \in N}\) cannot converge to \((0, 0),\) since each \(\Gamma_{jr}\) has an orbit which intersects \(\partial B_e.\) Therefore \(\Gamma = \lim_{j \to \infty} \Gamma_{jr}\) contains a nonempty set of orbits of (3.1). In addition, \(\Gamma\) is connected and extends to \(\partial B_{(1/2)\kappa}.\) Since the periods of orbits on \(\Gamma_{jr}\) approach \(2\pi/\beta_{jr}\) as the orbits converge to \((0, \lambda_{jr}),\) for each \(j,\) the virtual periods of orbits on \(\Gamma\) approach \(2\pi/\beta = \lim_{j \to \infty} \tau = \lim_{j \to \infty} \tau_{jr}\) as the orbits converge to \((0, 0).\) Thus \(\Gamma\) bifurcates from \((0, 0)\) starting with virtual period \(2\pi/\beta.\)

**REFERENCES**


