FINAL STATE SENSITIVITY: AN OBSTRUCTION TO PREDICTABILITY

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It is shown that nonlinear systems with multiple attractors commonly require very accurate initial conditions for the reliable prediction of final states. A scaling exponent for the final-state-uncertain phase space volume dependence on uncertainty in initial conditions is defined and related to the fractal dimension of basin boundaries.

Typical nonlinear dynamical systems may have more than one possible time-asymptotic final state. In such cases the final state that is eventually reached depends on the initial state of the system. In this letter we wish to consider the extent to which uncertainty in initial conditions leads to uncertainty in the final state. To orient the discussion, consider the simple two-dimensional phase space diagram schematically depicted in fig. 1. There are two possible final states ("attractors") denoted A and B. Initial conditions on one side of the boundary, \( \Sigma \), eventually asymptote to B, while those on the other side of \( \Sigma \) eventually go to A. The region to the left (right) of \( \Sigma \) is the "basin of attraction" for attractor A (or B, respectively) and \( \Sigma \) is the "basin boundary". If the initial conditions are uncertain by an amount \( \varepsilon \), then (cf. fig. 1), for those initial conditions within \( \varepsilon \) of the boundary, we cannot say a priori to which attractor the orbit eventually tends. For example, in fig. 1, 1 and 2 represent two initial conditions with an uncertainty \( \varepsilon \). The orbit generated by initial condition 1 is attracted to attractor B. Initial condition 2, however, is uncertain in the sense that the orbit generated by 2 may be attracted either to A or to B. In particular, consider the fraction of the uncertain phase space volume within the rectangle shown and denote this fraction \( f \). For the case shown in fig. 1, we clearly have \( f \sim \varepsilon \). It is one of the main points of this letter that, from the point of view of prediction, much worse scalings of \( f \) with \( \varepsilon \) frequently occur in nonlinear dynamics. In particular, if

\[ f \sim \varepsilon^\alpha, \]

with \( \alpha < 1 \), we shall say that there is \textit{final state sensitivity}. In fact, \( \alpha \) substantially less than one is, we believe, fairly common. In such a case, a substantial improvement in the initial condition uncertainty, \( \varepsilon \), yields only a relatively small decrease in the uncertainty of the final state as measured by \( f \).

While \( \alpha \) is equal to one for simple basin boundaries, such as that depicted in fig. 1, highly convoluted boundaries with noninteger (fractal) dimension also occur. We use here the capacity definition of dimension \[1\],

\[ d = \lim_{\delta \to 0} \frac{\ln N(\delta)}{\ln(1/\delta)}, \]

\textit{Fig. 1.} A region of phase space divided by the basin boundary \( \Sigma \) into basins of attraction for the two attractors A and B. 1 and 2 represent two initial conditions with uncertainty \( \varepsilon \).
where, if the dimensionality of the relevant phase space is $D$, then $N(\delta)$ is the minimum number of $D$ dimensional cubes of side $\delta$ needed to cover the basin boundary. In general, since the basin boundary divides the phase space, its dimension $d$ must satisfy $d \geq D - 1$. It can be proven \footnote{For the purposes of the proof of eq. (3) and for those of the present letter, eq. (1) should be regarded as shorthand for $\alpha = \lim_{e \to 0} \ln f/\ln e$. In addition, we note that, if the basins are unbounded regions, then we restrict attention to a bounded region of phase space (e.g., the rectangular region of fig. 1) for the purposes of calculating $f$ in (1) and $N(\delta)$ in (2).} \footnote{The Cantor set structure of the basin boundary for eqs. (4) is due to the presence of a horseshoe in the dynamics \cite{5}. In fact, eqs. (4) were chosen because they are a particularly straightforward example of this. (This point will be extensively discussed in ref. \cite{9}).} \footnote{Fractal basin boundaries which are curves (unlike that for fig. 2) have been discussed by Grebogi et al. \cite{7}.} \footnote{Fig. 2. Basins of attraction for eqs. (4). Arrows denote the two attractors.} that the following relation between the index $\alpha$ and the basin boundary dimension holds

$$\alpha = D - d.$$ 

(3)

For a simple boundary, such as that depicted in fig. 1, we have $d = D - 1$, and eq. (3) then gives $\alpha = 1$, as expected. For a fractal basin boundary, $d > D - 1$, and eq. (3) gives $\alpha < 1$ (i.e., there is final state sensitivity). To see heuristically how (3) comes about from (2), set the cube edge $\delta$ equal to the initial condition uncertainty $\varepsilon$. The volume of the uncertain region of phase space will be of the order of the total volume of all the $N(\varepsilon) D$-dimensional cubes of side $\varepsilon$ needed to cover the basin boundary. Since the volume of one of these $D$-dimensional cubes is $\varepsilon^D$, the uncertain volume of phase space is of the order of $e^{D N(\varepsilon)}$. Noting that $N(\varepsilon) \sim \varepsilon^{-d}$ satisfies (2), we use this to estimate the uncertain phase space volume as $e^{D N(\varepsilon)} \sim \varepsilon^{D-d}$, which is the result predicted by eqs. (1) and (3).

We now illustrate the above with a concrete example. Subsequently, we argue that the demonstrated phenomena occur commonly in nonlinear dynamics (e.g., in the Lorenz system \cite{3} and in experiments on fluids such as those of Bergé and Dubois \cite{4}).

We consider the two-dimensional map

$$\begin{align*}
\theta_{n+1} &= \theta_n + a \sin 2\theta_n - b \sin 4\theta_n - x_n \sin \theta_n , \\
x_{n+1} &= -J_0 \cos \theta_n ,
\end{align*}$$

(4a)

(4b)

where $\theta$ and $\theta + 2\pi$ are identified as equivalent. This map has two fixed points ($\theta, x) = (0, -J_0)$, and $(\pi, J_0)$, which are attracting for $|1 + 2a - 4b| < 1$. Numerical experiments on eqs. (4) with different sets of parameters have been performed. Here, as an example, we report results for $J_0 = 0.3$, $a = 1.32$, and $b = 0.90$. For these parameters we find numerically that the only attractors are the fixed points $(0, -0.3)$ and $(\pi, 0.3)$. Fig. 2 shows a computer generated picture of the basins of attraction for the two fixed point attractors. This figure is constructed using a $256 \times 256$ grid of initial conditions. Each initial condition is iterated until it is close to one of the two attractors (100 iterates of the map is always sufficient to accomplish this). If an orbit goes to the attractor at $\theta = 0$, a black dot is plotted at the corresponding initial condition. If the orbit goes to the other attractor, no dot is plotted. Thus the black and blank regions are essentially pictures of the basins of attractions for the two attractors to the accuracy of the grid used and of the computer plotter. (Due to the symmetry of the map we have only shown $0 \leq \theta \leq \pi$ in fig. 2). Fine scale structure in the basins of attraction is evident. This is a consequence of the Cantor set nature of the basin boundary \footnote{Cantor set basin boundaries due to horseshoes are not curves (i.e., they cannot be represented as $x = x(u), \theta = \theta(u)$ with $x(u)$ and $\theta(u)$ continuous functions of $u$), and have been known for a long time (see, for example ref. \cite{6}). Fractal basin boundaries which are curves (unlike that for fig. 2) have been discussed by Grebogi et al. \cite{7}.}. In fact, magnifications of the basin boundary show that, as we examine it on a smaller and smaller scale, it continues to have structure.
We now wish to explore the consequences for prediction of this infinitely fine scaled structure. To do this, consider an initial condition \((\theta_0, x_0)\). We ask, what is the effect of a small change \(e\) in the \(x\)-coordinate? Thus we iterate the initial conditions \((\theta_0, x_0), (\theta_0, x_0 + e),\) and \((\theta_0, x_0 - e)\) until they approach one of the attractors. If either or both of the perturbed initial conditions yield orbits which do not approach the same attractor as the unperturbed initial condition, we say that \((\theta_0, x_0)\) is uncertain. Now say that we randomly choose a large number of initial conditions in the rectangle shown in fig. 2, and let \(f\) denote the fraction of these which we find to be uncertain. From the definitions of \(\bar{f}\) and \(f\) (\(f\) is the fraction of uncertain phase space volume) we expect that \(f\) is approximately proportional to \(\bar{f}\), and hence \(f \sim e^{D-d}\). Fig. 3 shows results from a set of numerical experiments on the scaling of \(f\) with \(e\). In generating this figure, 8192 randomly chosen initial conditions were used for each value of \(e\). The statistical error in the number \(N'\) of uncertain initial conditions at each \(e\) was estimated to be \(\frac{\sqrt{N'}}{N'}\), and this was used in determining the error bars shown in the figure. Linear dependence of \(\log f\) with \(\log e\) is evident, thus indicating an approximate power law dependence, eq. (1). We find from fig. 3 that \(\alpha \approx 0.2\). Thus, from (3), the dimension of the basin boundary is \(d \approx 1.8\). Note, in particular, the numerical values that result (cf. fig. 3):

\[
\begin{array}{c|c|c}
\hline
\epsilon & f & \bar{f} \\
\hline
0.125 & 0.59 & 0.59 \\
0.002 & 0.26 & 0.26 \\
3 \times 10^{-5} & 0.12 & 0.12 \\
\hline
\end{array}
\]

The implication is that extraordinarily high accuracy of initial conditions may sometimes be necessary for the reliable prediction of the eventual final state. Furthermore, the example just discussed is by no means extreme or unusual. As evidence for this we cite two examples:

1. The Lorenz system. In his pioneering study, Lorenz [3] examined a system of three first-order ordinary differential equations modeling the Bénard instability. In this model a chaotic attractor occurs when the Rayleigh number \(r\) exceeds a critical value \(r_c \approx 24.06\). For \(1 < r < r_c\) there is no chaotic attractor, but there remain two nonchaotic attractors, one representing steady counterclockwise convective flow and the other representing steady clockwise convective flow. Based on the results of ref. [8], it follows that [9], for \(r_c > r > r_c' \approx 13.93\), the Lorenz system has final state sensitivity (i.e., \(\alpha < 1\)) with respect to the two steady convective attractors. In addition, the sensitivity can be much more severe than in our numerical example, figs. 2 and 3. For example, using ref. [10] we can crudely estimate [9] \(\alpha \approx 0.1\) at \(r_c - r = 4\), \(\alpha \approx 10^{-2}\) at \(r_c - r = 1.6\), and \(\alpha \approx 10^{-3}\) at \(r_c - r = 0.8\), with \(\alpha \to 0\) as \(r \to r_c\).

2. The experiments of Bergé and Dubois [4]. These authors have performed experiments on the Bénard instability in a low aspect ratio rectangular cell for high Rayleigh number and high Prandtl number. They observe that the system can have multiple attractors, and that a rather long chaotic transient [10,11] exists before the system settles into one of the attractors. Since the system evolution during the transient phase depends strongly on initial conditions, it is to be expected that the final state will also, and that \(\alpha < 1\) in eq. (1) will apply. At somewhat lower Rayleigh number (450 \(\geq r > 200\)) different stable attractors still simultaneously coexist but long chaotic transients do not occur. Even in this range it is probable that final state sensitivity will occur (e.g., in our example, fig. 2, the average decay time to the vicinity of one of the attractors is only about five iterates).

In conclusion, the notion of final state sensitivity
has been introduced, and its implications for prediction have been illustrated by a numerical experiment and by suitable interpretation of previous theoretical [8] and experimental [4] work.

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References


