CHAOTIC BEHAVIOR OF MULTIDIMENSIONAL DIFFERENCE EQUATIONS

by

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In the first half of this paper we present results on chaotic behavior in multidimensional spaces. In the second half we discuss the problem of determining the dimension of an attractor. We believe it is possible to predict the dimension of the invariant attractor, at least generically, from a knowledge of the Lyapunov numbers of the map. Computer generated illustrations are given in evidence.

In a series of papers A. N. Sharkovskii investigated chaotic behavior of one-dimensional maps. The following theorem is in this spirit. See [1] for a proof.

Theorem 1. (Period three implies chaos). Consider the first order difference equation

(1) $$x_{k+1} = f(x_k) , \qquad k = 0,1,2,\ldots$$

where $x_k \in R$, $f:R \to R$ is continuous. Let J be an interval and let $f:J \to J$ be continuous. If there exists a point of period 3 for (1) then

(i) There is an uncountable set $S \subset J$ (containing no periodic points) satisfying:

(a) For every $p,q \in S$ with $p \neq q$, $\limsup\limits_{n\to\infty} |f^n(p) - f^n(q)| > 0$,

(b) For every $p \in S$ and periodic point $q \in J$ $\limsup\limits_{n\to\infty} |f^n(p) - f^n(q)| > 0$,

(c) There exists an uncountable set $S_0 \subset S$ such that $\liminf\limits_{n\to\infty} |f^n(p) - f^n(q)| = 0$ for all $p,q \in S_0$.

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Although "chaotic" behavior was originally observed in the study of a hydrodynamical system [2] this phenomenon has drawn considerable attention from mathematical ecologists. There are many situations in which population growth is a discrete process, and where the appropriate models are difference equations relating the population in generation $\tau + 1$, $N_{\tau+1}$, to that in generation τ, N_τ. Biological examples are provided by many temperate zone arthropod populations, with one short lived adult generation each year.

For example, among the many density dependent forms which have been proposed as discrete analogs of the logistic equation are the quadratically nonlinear equation:

(2) $$N_{k+1} = N_k \left[1 + r\left(1 - {}^{N_k}/_K\right)\right]$$

and the equation

(3) $$N_{k+1} = N_k \exp\left[r\left(1 - {}^{N_k}/_K\right)\right].$$

(For a complete discussion of equations (2) and (3), as well as a cataloguing of other related models, see [3] and [4].)

It may be instructive to examine in greater detail the behavior of solutions of equation (2). We first make the preliminary change of variables:

$$x_k = {}^1/_K \left({}^{a-1}/_a\right) N_k, \qquad a = 1 + r$$

which reduces (2) to:

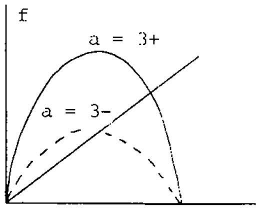
(4) $$x_{k+1} = ax_k(1-x_k)$$

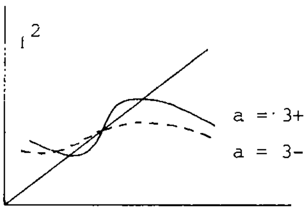
We restrict ourselves to the parameter values $o \le a \le 4$, since for such values $f(x) = ax(1-x)\colon [0,1] \to [0,1]$. Equation (4) is the most frequently studied example of a chaotic scalar difference equation. (See [1], for example).

For $1 < a \le 4$, f has the nontrivial fixed point $z = {}^{a-1}/_a$. When $1 < a < 3$, $-1 < f'(z) < o$ and we have a locally stable fixed point. As a increases beyond 3, the behavior is determined by examining $f^2(x) = f(f(x))$.

We easily compute:

$$\frac{d}{dx} f'(x)\Big|_{x=z} = f'(f(z))f'(z) = [f'(z)]^2$$



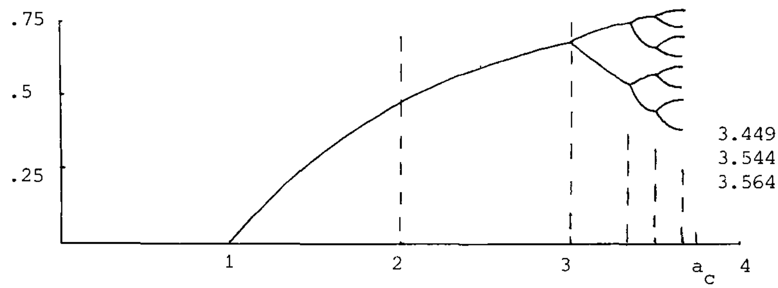


As the fixed point z remains stable with a→3- the map f^2 has a slope less than, but approaching, +1. It follows that its graph can only cross the 45° line once in a neighborhood of z. But as a increases past 3, the slope of f^2 exceeds 1, so that the graph of f^2 must cross the 45° line three times in the neighborhood of z. Exactly at the parameter value where the fixed point becomes unstable (a = 3) we find a new (and locally stable) solution of period 2. Thus there is a bifurcation of the fixed point into two locally stable points of period 2, between which the population oscillates.

As a increases these two points (2-cycle) in turn become unstable and bifurcate to give four locally stable fixed points of period 4. In this way there arises, by successive bifurcations, an infinite collection of stable cycles, of period 2^n.

This sequence of stable cycles of period 2^n converges on a limiting value of a, say a_c. ($a_c \simeq 3.570$). For the difference equation (4), it can be shown by direct computation that there exists a point of period 3 for a > 3.83. Thus there will be chaos in this regime.

Similarly, if we consider the map f^2, this will have a point of period 3 for a > 3.67. Thus f^2 (and hence f) will be chaotic in this region. By considering the 3 cycles of f^n as $n \to \infty$ we find that as n increases the value of a for which Theorem 1 implies chaos decreases monotonically and approaches a_c.



Difference schemes describing population growth are by no means restricted to single species models. There are prey-predator systems and competitive systems where generations do not overlap.

In an interesting paper, Beddington, Free and Lawton [5] investigate the particular host parasite system.

$$(5) \qquad H_{k+1} = H_k \exp[r(1 - {}^{H_k}/_{K}) - aP_k],$$

$$P_{k+1} = \alpha H_k\,[1 - \exp(-aP_k)] \qquad k = 0, 1, 2, \ldots..$$

where r, a, α, $K \varepsilon R^+$. Numerical studies show regions of stable points, of stable limit cycles and of chaos. The onset of chaotic behavior takes place at lower r values than for the corresponding single species model. This paper is illustrated by fascinating photographs of oscilloscope trajectories.

In another paper, Guckenheimer, Oster and Ipaktchi [6] discuss another discrete model (Leslie model). Here a single population has been divided into two age classes x and y:

$$(6) \qquad \begin{pmatrix} x \\ y \end{pmatrix}_{k+1} = \begin{pmatrix} m_1 & m_2 \\ s & o \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_k \;;$$

s is the fraction of individuals in the first age class x that survive to the second age class, $m_1(\cdot)$ and $m_2(\cdot)$ are the per capita birthrates for the two age classes. The density dependence of the birthrates is described by:

$$m_i(x,y) = b_i e^{-a(x+y)} \qquad i = 1, 2,$$

This difference scheme may be described by the map $f: R^2 \to R^2$ given by:

$$(7) \qquad (x_{k+1}, y_{k+1}) = f(x_k, y_k) = [(b_1 y_k + b_2 y_k)\, e^{-a(x_k+y_k)}, s x_k] \qquad i = 1, 2.$$

The coeffecients b_1 and b_2 give the maximum per capita reproductive rates for each age class. As in system (5) numerical investigation of the model shows that there are stable points, stable cycles, stable "invariant curves" or chaotic orbits, depending on the values of the various parameters. The authors offer a heuristic explanation of this behavior.

It is significant to note that the behavior of these multidimensional systems cannot be described or explained in terms of Theorem 1. The proof of Theorem 1 depends, in an essential way, upon a fixed point theorem which is only valid for real valued mappings. In fact, it is possible to give simple examples of difference equations on R^2 for which stable 3-cycles exist. Thus an alternate analysis is required for higher dimensional systems like (5) or (6). Numerous cases of such systems have been investigated by Mira and Gutowski in a long series of papers.

This problem was treated by F.R. Marotto, a former student of the first author, in [7]. Consider:

$$(8) \qquad x_k = f(x_k) \qquad f: R^n \to R^n, \text{ f continuous}.$$

In order to illustrate his ideas let us consider the following analog of equation (4).

$$x_{k+1} = (ax_k + bx_{k-1})\ (1 - ax_k - bx_{k-1}).$$

If we set $y_k = x_{k+1}$ this may be written in the form of (8) with $n = 2$:

$$(9) \qquad \text{(a)}\ x_{k+1} = (ax_k + by_k)\ (1 - ax_k - by_k)$$

$$\text{(b)}\ y_{k+1} = x_k.$$

Since we are interested in only the positive solutions of (9) we restrict our parameters to the region R of the (a,b) parameter space:

$$R = \{(a,b): a\geq o,\ b\geq o,\ a+b\leq 4\}$$

In this case:

$$D = \{(x,y) : 0\leq x,y\leq 0,25\}$$

is invariant. The dynamics of this difference scheme will be strongly dependent upon the eigenvalues of the Jacobian of the right hand side of (9), evaluated at the fixed points. There are two such fixed points: $x_k = y_k = 0$ (trivial fixed point and for $a+b\geq 1$, $x_k = y_k = \frac{a+b-1}{a+b}$.

A simple calculation shows that:

$$Df(x,y) = \begin{bmatrix} a - 2a(ax+by) & b - 2b(ax+by) \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of (9) at $z = \left(\frac{a+b-1}{a+b}, \frac{a+b-1}{a+b}\right)$ satisfy

$$\lambda^2 + A\lambda + B = 0$$

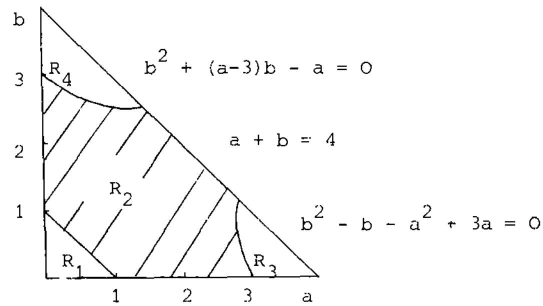
where

$$A = \frac{a(a+b-2)}{a+b}, \quad B = \frac{b(a+b-2)}{a+b}.$$

It is easily checked that for a + b near 1, the point z is stable. However, moving away from this line there are two ways in which z can become unstable:

(i) both eigenvalues are real and one of them exceeds one in absolute value (R_3), or

(ii) eigenvalues are complex conjugates of one another, and both exceed one in norm. (R_4)



Definition: Let $f: R^n \to R^n$ be continuously differentiable in some ball $B_r(z)$ about the fixed point z. We will say that z is a repelling fixed point of f if all the eigenvalues of $Df(x)$ exceed 1 in absolute value for all $x \in B_r(z)$.

The repelling fixed point z will be called a snap-back repeller if there exists $x_0 \in B_r(z)$, $x_0 \neq z$, and an integer M such that:

$$f^M(x_0) = z$$

and

$$|Df^M(x_0)| \neq 0$$

Marotto proved the following result [7].

Theorem 2. Snap-back repellers imply chaos in R^n. More precisely, suppose z is a snap-back repeller for f . Then

(i) there is an integer N such that for every $k \geq N$ there exists a periodic point in $B_r(z)$ having period k,

(ii) there is an uncountable set $S \subset B_r(z)$ containing no periodic points satisfying:

(a) For every $p,q \in S$ with $p \neq q$

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0,$$

(b) For every $p \in S$ and periodic point $q \in J$,

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0,$$

(c) There is an uncountable $S_o \subset S$ such that for all $p, q \in S_o$,

$$\liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0.$$

A sketch of the proof of Theorem 2 can also be found in [8].

It should be noted that if $f:R \to R$ is continuously differentiable, then the existence of a snap-back repeller for f is equivalent to the existence of a point of period 3, so that Theorem 2 is a true generalization of Theorem 1.

The arguments used to establish Theorem 2 are reminiscent of Smale's famous "horseshoe" example [9]. Smale showed that for a conditionally stable fixed point of a diffeomorphism the assumption of a transverse homoclinic orbit implies the existence of an infinite number of periodic points of different periods. In our case, the snap-back repeller plays the role of the homoclinic orbit, while the condition $|Df^M(x_o)| \neq 0$ can be viewed as a transversality condition.

Theorem 2 provides sufficient conditions for establishing chaotic behavior in multidimensional difference schemes such as (5), (6) and (9). Note, however, that Theorem 2 may be applied to (9) only in the region R_4, where the equation possesses a snap-back repeller. In order to analyze behavior in R_3 we must employ alternate methods. These are based on the observation that, for small b, equation 9(a) can be viewed as a perturbation of

$$x_k = ax_k(1 - ax_k)$$

for which chaotic behavior has already been demonstrated.

Consider

$$(10) \qquad x_{k+1} = f(x_k, bx_{k-1}),$$

$f:R^2 \to R$, f continuously differentiable. If we rewrite (10) in system form,

$$\text{(a)} \quad x_{k+1} = f(x_k, by_k)$$

$$\text{(b)} \quad y_{k+1} = x_k$$

then the fixed point (x_o, x_0) will be said to be neutrally stable if at least one eigenvalue of the associated linearized mapping at (x_o, x_o) has norm equal to 1.

The following Lemma 3 and Theorem 4 are modifications of a private communication from F. Marotto to the first author.

Lemma 3 Suppose (10) has a fixed point x_o which is not neutrally stable when $b = 0$. Then

(i) there exists $a, c \varepsilon \{R, \infty\}$ with $a < o$, $c > o$ such that (10) has a fixed point $x(b)$ with the same stability as x_o, for all $b \varepsilon (a,c)$,

(ii) $x(b)$ is a uniquely defined continuous function of b for $b \varepsilon (a,c)$ with $x(o) = x_o$, and

(iii) if $-\infty < a$ (or $c < \infty$) then as $b \to a$ (as $b \to c$) either $x(b)$ is unbounded or $x(b)$ becomes neutrally stable.

Sketch of proof. Define $g(x,b) = x - f(x,bx)$.

Then

$$g(x_o, o) = x_o - f(x_o, o) = 0$$

while

$$g_1(x,b) = 1 - f_1(x,bx) - bf_2\ (x,bx)$$

so that

$$g_1(x_o, o) = 1 - f_1(x_o, o) \neq 0$$

since x_o is not neutrally stable. It follows from the Implicit Function Theorem that there exists a unique continuous function $x(b)$ such that

$$g\big(x(b),\ b\big) = 0$$

for b sufficiently small. Thus $x(b) = f(x(b),\ b\ x(b)$.

Standard continuation arguments as well as continuous dependence of eigenvalues of a matrix upon its entries imply the remaining conclusion.

Theorem 4. Suppose 11(a) has a snap-back repeller when $b = 0$. Then (10) has a structurally stable homoclinic orbit for all $|b|<\varepsilon$ for some $\varepsilon>0$. In particular, for $|b|<\varepsilon$ (10) is chaotic.

Sketch of proof. Consider the reduced system,

$$\text{(12)} \qquad \begin{aligned} \tilde{x} &= f(x,o) \\ \tilde{y} &= x \end{aligned}$$

obtained from (11) by setting $b = 0$. Consider the curve C described by

$$C = \{(f(t,o),\ t) : t\varepsilon R\}.$$

The curve C is invariant under (12) because for any $(x,y)\varepsilon$ C,

$$(\tilde{x},\tilde{y}) = (f(x,o),\ x)\varepsilon C.$$

Note that the curve C is simply the graph $x = f(y,o)$.

Now if $x_{k+1} = f(x_k,o)$ has a snap-back repeller then by definition there exists a point z satisfying $z = f(z,o)$, $|f_1(z,o)| > 1$. Note that (z,z) is a fixed point of (12). The eigenvalues of the Jacobian matrix of (12) at (z,z) satisfy:

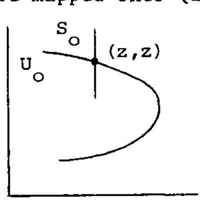
$$\det\begin{bmatrix} f_1(z,o) - \lambda & o \\ 1 & -\lambda \end{bmatrix} = (f_1(z,o) - \lambda)(-\lambda) = o.$$

Thus, $\lambda_1 = o$ is associated with the stable manifold S_o and $|\lambda_2| = |f_1(z,o)| > 1$ is associated with the unstable manifold U_o. In fact, in a local neighborhood of (z,z) these manifolds can be identified precisely.

S_o. Since S_o is composed of those points which approach (z,z) under iteration of (12) we see that locally

$$S_o = \{(x,y) : x = z\}.$$

In fact, all points in S_o are mapped onto (z,z) precisely.



U_o. The curve C is U_o. As noted above C is an invariant manifold in the x,y plane. Moreover the points of C are mapped away from $(z,z)\varepsilon C$ under (12) (because (z,z) is a repelling fixed point).

Now since z is a snap-back repeller there must exist a homoclinic point at the intersection of S_o and U_o (i.e., the orbit in R^2 around the snap-back repeller must be a homoclinic orbit.).

By Lemma 3, for $|b| < \varepsilon$, there exists $(x(b), y(b)) = z_b$ a solution of (11) such that $(x(o), y(o)) = (z,z)$, z_b depends continuously upon b, and z_b is a fixed point of (11) with the same stability as (z,z). The structural stability of the homoclinic orbit of (12) now implies the existence of a homoclinic orbit of (11) for b sufficiently small.

Having proved Theorem 4 we can now demonstrate chaotic behavior in a manner similar to Smale's horseshoe arguments.

Remark: The small perturbation arguments outlined above can be adapted to several other situations. For instance, we could just as easily have considered the equation,

$$x_{k+1} = f(x_k, b_1 x_{k-1}, b_2 x_{k-2}, \ldots, b_m x_{k-m}).$$

An interesting case is the system described by $f,g: R^2 \to R$,

$$(13) \qquad \begin{aligned} x_{k+1} &= f(x_k, by_k) \\ y_{k+1} &= g(cx_k, y_k). \end{aligned}$$

Consider the uncoupled system

$$(14) \qquad \begin{aligned} &(a) \quad x_{k+1} = f(x_k, o) \\ &(b) \quad y_{k+1} = g(o, y_k). \end{aligned}$$

If (14)a has a snap-back repeller and 14(b) has an unstable fixed point then (13) has a snap-back repeller for $|b|$, $|c|$ sufficiently small. If 14(a) has a snap-back repeller and 14(b) has a stable fixed point then (13) has a structurally stable homoclinic orbit for $|b|$, $|c|$ sufficiently small. These ideas can be applied to the (competitive) system

$$(15) \qquad x_{n+1} = \lambda x_n [1 + a(x_n + \alpha y_n)]^{-b}$$

$$y_{n+1} = \lambda' y_n [1 + a'(y_n + \beta x_n)]^{-b'}.$$

For $\alpha = \beta = o$, each of the possibilities enumerated above can be obtained for selected parameter values, and thus we can conclude the chaotic behavior of the full system (15), which has been observed numerically by Hassell and Comins [10].

In our examination of system (9) we found that there were two chaotic regions, R_3 and R_4. Our analysis of the system for parameters chosen from R_3 indicated that the chaotic set is one dimensional (in fact, it is a perturbation of the curve $x = ay(1-ay)$) while in the region R_4 it appears that the chaotic set is two dimensional. In general, given a difference scheme $x_{k+1}=f(x_k)$, $f:R^n \to R^n$, is there any way to decide, a priori, what the dimension of the chaotic set will be?

In order to formulate a conjecture on this question it is first necessary for us to formulate a notion of dimension. Intuitively, the dimension of a space indicates the amount of information required to specify a location. In practice, positions are given approximately to within ε. For a space X, let $N(\varepsilon)$ be the minimum number of points that can be chosen so that the ε balls centered at these points cover the space. The dimension of X tells us how fast $N(\varepsilon)$ grows as ε shrinks to o. If the set is n dimensional we expect $N(\varepsilon) \approx \varepsilon^{-n}$. Define

$$\dim X = \lim_{\varepsilon \to o} - \frac{\ln N(\varepsilon)}{\ln \varepsilon}$$

whenever the limit exists.

This is metric space dimension (Hausdorff dimension). It is easy to give examples where dim X is not an integer (see work of Mandelbrot). This fact is an interesting sidelight which should not be distracting.

At least two other concepts of dimension are commonly used. One is linear dimension (vector space dimension) of a vector space, the maximal number of linearly independent vectors. This is not applicable to our work since the chaotic attracting set will not be a vector space. The second notion is that of topological dimension. The topological dimension is n if there is a cover of open sets such that for every refinement there are points that are in at least n+1 open sets of the refinement. One of the most striking aspects of this definition is how far removed it is from any idea of measurement. For instance, if $F:X \to X$ where X,Y are compact metric spaces, then top dimension is not well behaved since we can have $\dim F(X) > \dim X$: you can map an interval continuously onto a square.

For Lipschitz mappings the metric space dimension is better behaved. It can be shown to satisfy

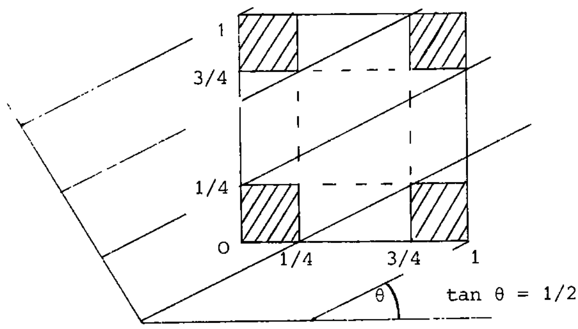
$$\dim F(X) \le \dim X$$

$$\dim (X_1+X_2) = \dim X_1 + \dim X_2 .$$

Remark: For Lipschitz maps we might define an integer valued dimension as follows. If there is a Lipschitz map $F:X \to R^n$ such that F(X) contains a ball in R^n we say X is at least n dimensional, and the dimension of X would then be the largest such number. We conjecture that for spaces whose Hausdorf dimension is an integer, these two definitions are equal.

The following is an example of a set in R^2 with metric space dimension 1.

Example: A space with Hausdorff dimension 1. Let Y = the Cantor set obtained by removing open middle halves from the interval [0,1], let X = Y x Y



Then X has Hausdorff dimension 1. Also there is a linear projection of $R^2 \to R^1$ which takes X onto an interval.

In order to formulate our conjecture, we examined numerically a class of difference equations of the form

(16) $$x_k = f(x_{k-1})$$

where f: $B \to B^o$, B compact, $f(B) \subset \mathrm{Int} B$.

In particular, we examined the family of equations

(17) $$x_{k+1} = 2x_k \bmod 1$$
$$y_{k+1} = \lambda_1 y_k + p(x_k)$$

and

(18) $$x_{k+1} = 2x_k \bmod 1$$
$$y_{k+1} = \lambda_1 y_k + p(x_k)$$
$$z_{k+1} = \lambda_2 z_k + q(x_k)$$

where p,q are periodic with period 1. For (17)

$$B = \{(x,y): \ o \le x \le 1, \frac{1}{1-\lambda_1} P_{min} \le y \le \frac{1}{1-\lambda_1} P_{max}\}$$

while for (18)

$$B = \{(x,y,z): 0 < x < 1, \frac{1}{1-\lambda_1} p_{min} \le y \le \frac{1}{1-\lambda_1} p_{max}, \frac{1}{1-\lambda_2} q_{min} \le z \le \frac{1}{1-\lambda_2} q_{max}$$

These systems were chosen because of the ease with which it is possible to compute

$$\mathrm{Det}\ Df = 2\lambda_1$$

for equation (17), and

$$\mathrm{Det}\ Df = 2\lambda_1\lambda_2$$

for equation (18).

<u>Notation</u>: Let A be an n×n matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $|\lambda_1| \ge |\lambda_2| \ge \ldots \ge |\lambda_n|$. Write

$$\mathrm{Det}_j A = |\lambda_1\lambda_2 \ldots \lambda_j| \ .$$

We say a map f has <u>Lyapunov numbers</u> if

$$\delta_j = \lim_{k\to\infty}\{\mathrm{Det}_j(Df^k)\}^{1/k}$$

exists, and the Lyapunov numbers are $L_1 = \delta_1$, $L_j = \delta_j/\delta_{j-1}$ for $j = 2,3,\ldots$. Note that A has Lyapunov numbers $L_i = |\lambda_i|$.

For system (18) observe that the Lyapunov numbers are 2, λ_1 and λ_2.

For system (17) we found that if $2\lambda_1 > 1$ then the chaotic set appears 2 dimensional, if $2\lambda_1 \le 1$ then the chaotic set appears 1 dimensional. (See figures.)

For system (18) we found that if $2\lambda_1\lambda_2 > 1$ then the chaotic set appears 3 dimensional, if $2\lambda_1 > 1$ while $2\lambda_1\lambda_2 \le 1$ then the chaotic set is 2 dimensional. This relationship holds for "almost" every choice of functions p and q. For certain special choices, however, this relationship broke down.

For example, if

$$\lambda_1 = \lambda_2 \quad \text{and} \quad p = q$$

then $y = z$ and (18) degenerates into a 2 dimensional system, because of symmetry, regardless of the magnitude of λ_1 and λ_2.

Similarly, if $p(x) = \cos 8\pi x$ while $q(x) = \cos 4\pi x$ and $\lambda_1 = \lambda_2$ the system degenerated because $x_n = 2x_{n-1} \bmod 1$. These examples lead us to make the following conjecture.

Conjecture: Let $f\colon B \to B^0$, B compact and consider

$$(16) \qquad x_k = f(x_{k-1}).$$

Then for nearly every (in the generic sense) f satisfying

$$(19) \qquad \liminf_{m\to\infty} \{\mathrm{Det}_j\,(Df^m)\}^{1/m} > 1$$

the attracting limit set $\bigcap_{m=1}^{\infty} f^m(B)$ has metric dimension $\geq j$.

Thus far we have been able to develop a heuristic argument for the validity of this conjecture, but to date we have been unable to provide a rigorous proof.

The above conjecture concerns the integer part of the dimension. More specifically, assume f has Lyapunov numbers. Assume we can choose j such that $\delta_j > 1 > \delta_{j+1}$. Then we conjecture that generically the metric dimension $= j$ + fraction, where the fraction is given by

$$\frac{\log \delta_j}{|\log L_{j+1}|}$$

This latter number can be shown to be less than 1.

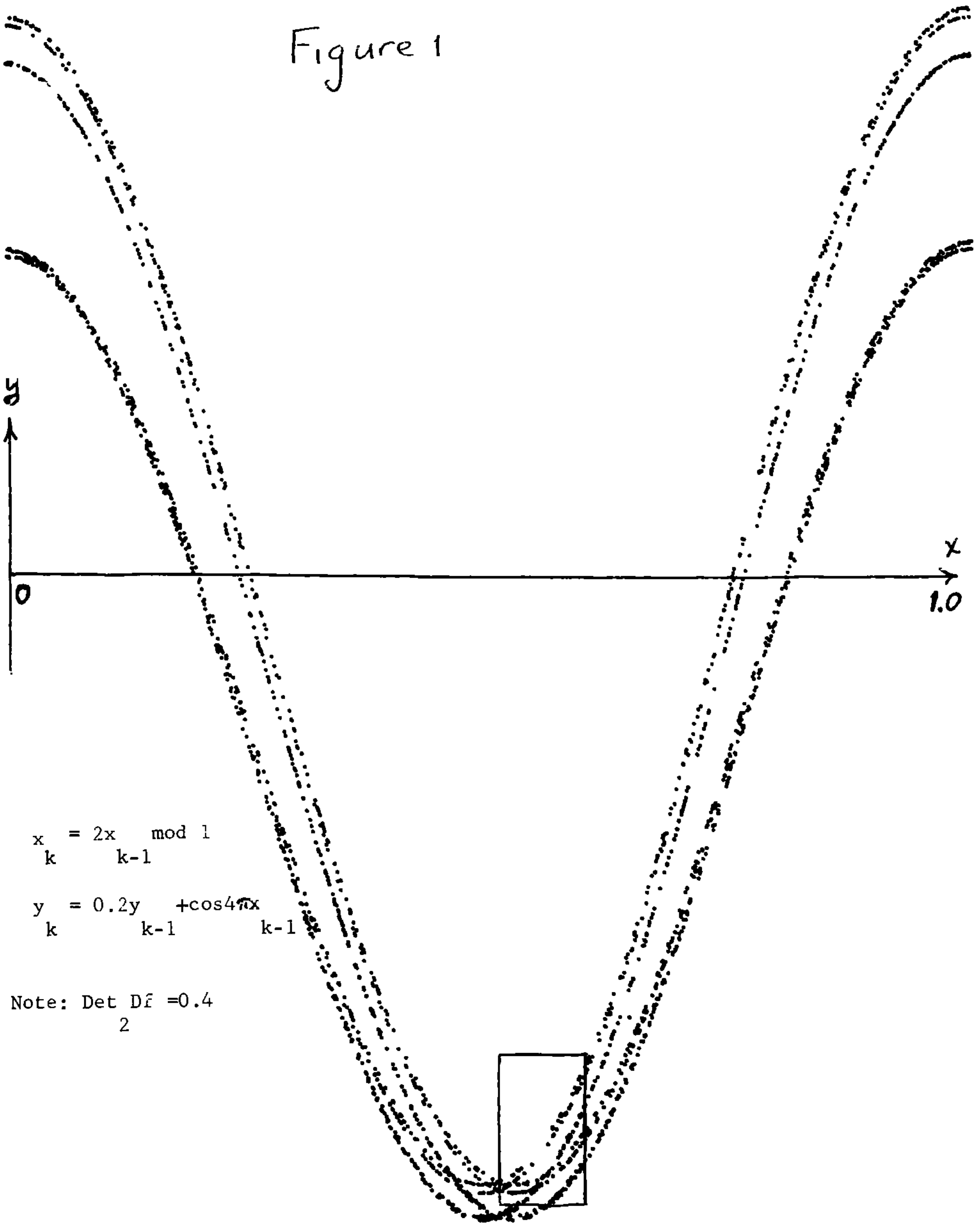


Figure 1
y
x
0
1.0
$x_k = 2x_{k-1} \bmod 1$
$y_k = 0.2y_{k-1} + \cos 4\pi x_{k-1}$
Note: Det Df_2 =0.4

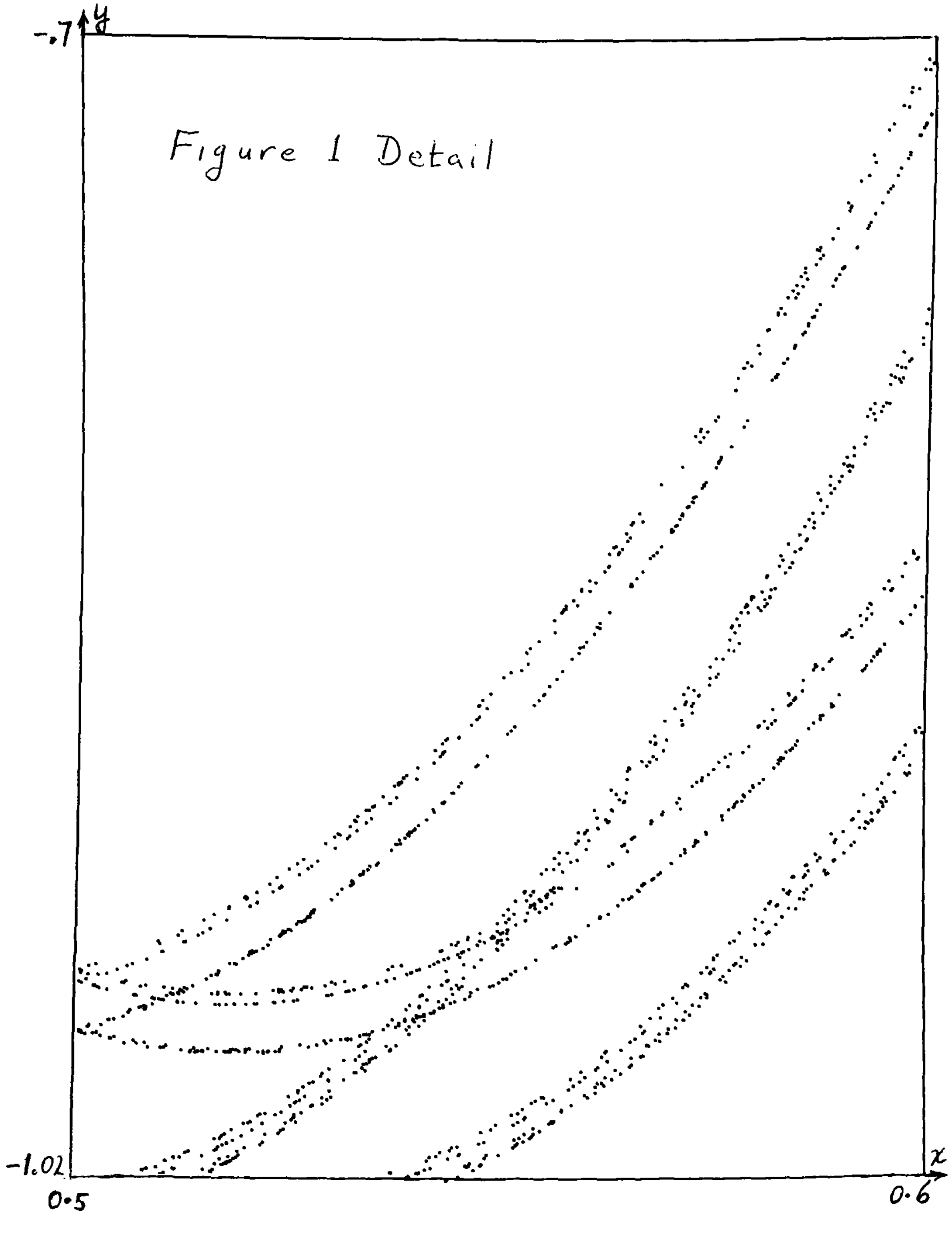
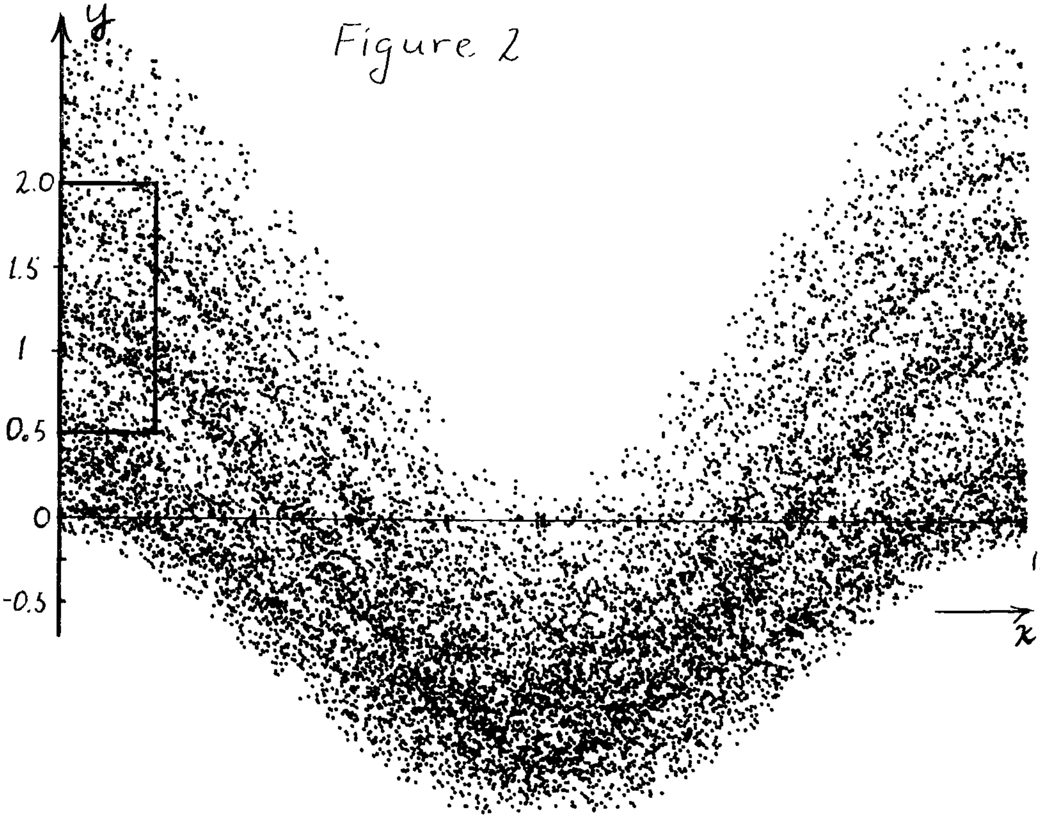


Figure 1 Detail
y
-.7
-1.02
x
0.5
0.6

Figure 2

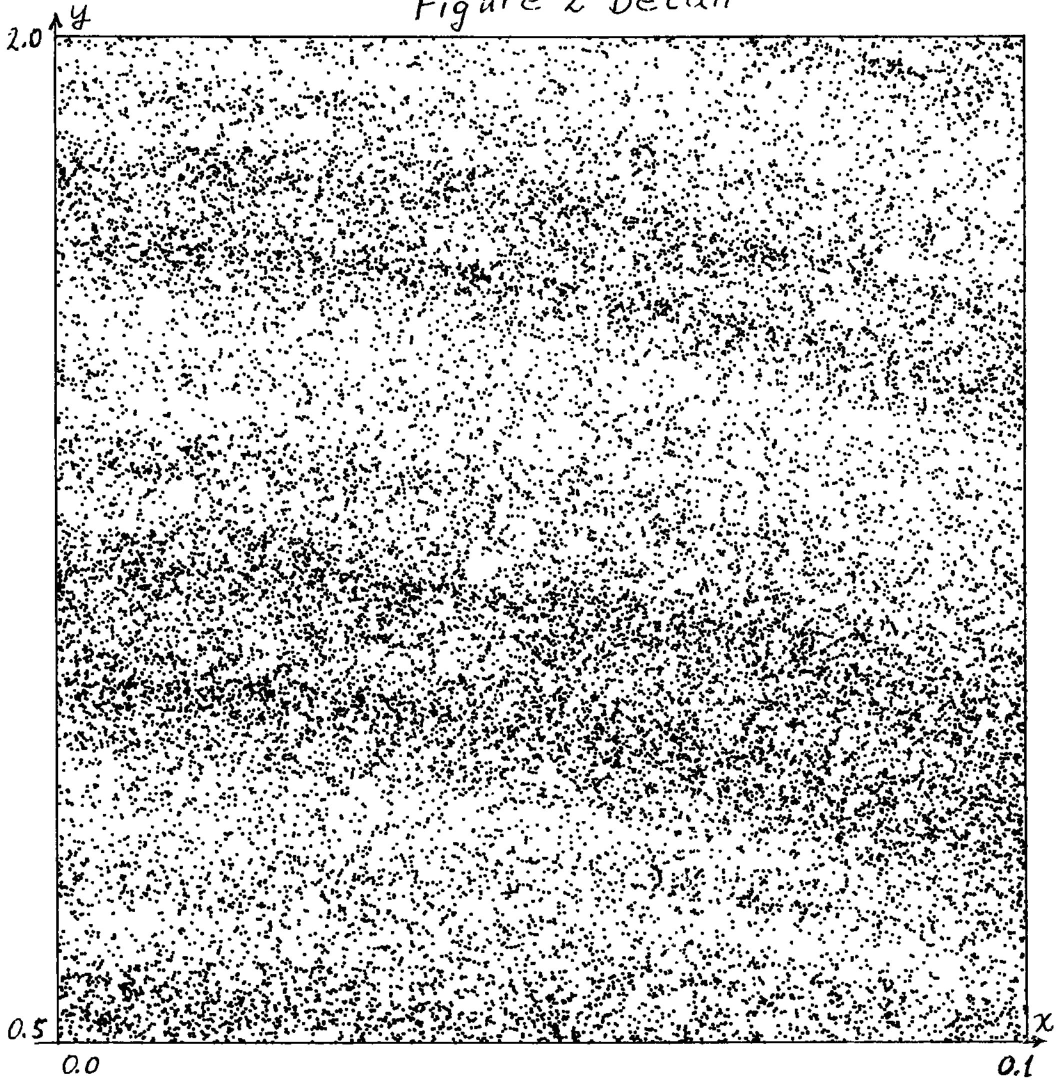


12000 points plotted

$$x_{k+1} = 2x_k \bmod 1$$

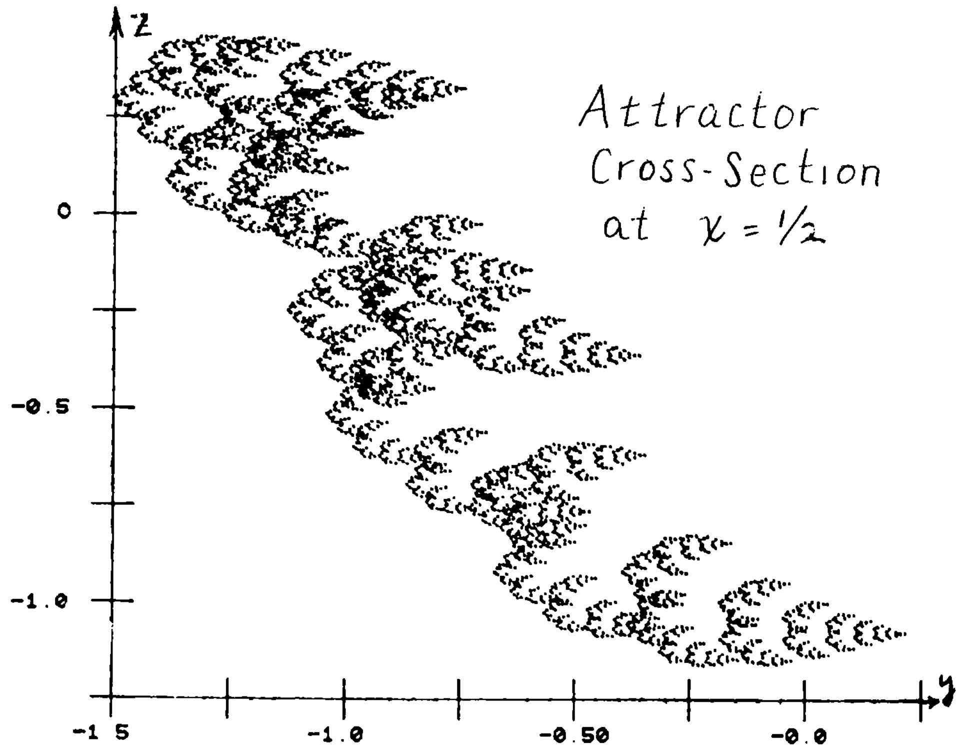
$$y_{k+1} = 2/3\, y_k + \cos(4\pi x_k)$$

Figure 2 Detail



20000 points plotted

Figure 3



16385 points plotted

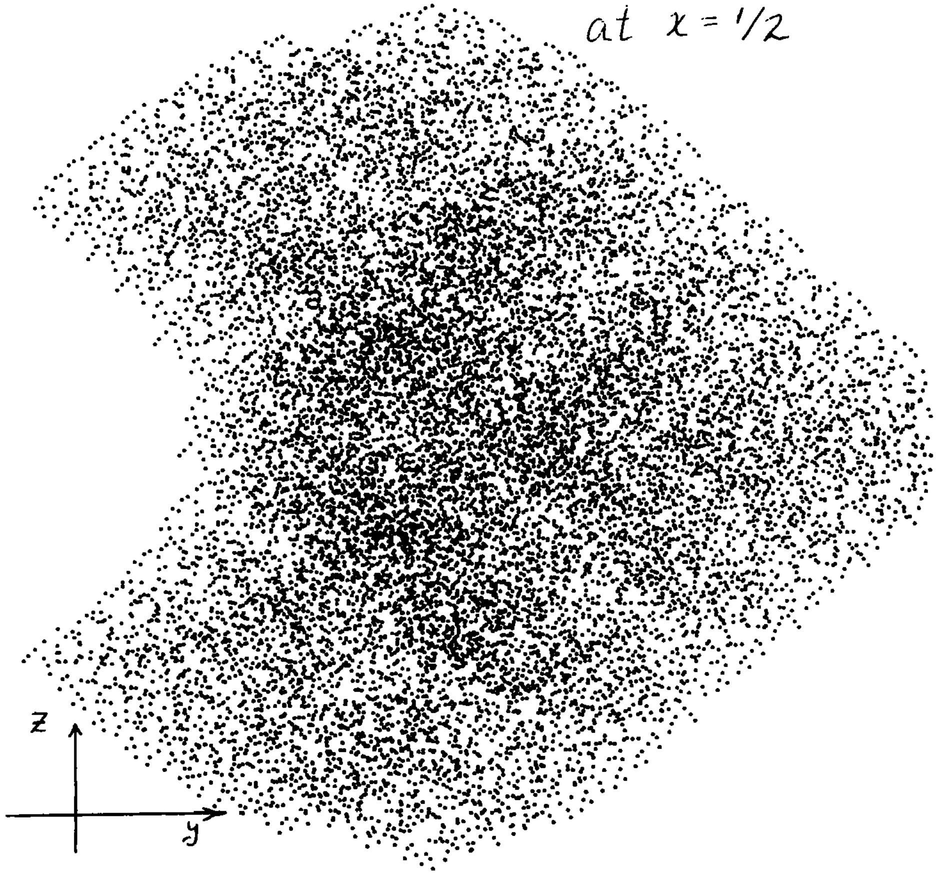
$\mathrm{Det}\ Df_3 = 8/9$

$$x_{k+1} = 2x_k \bmod 1$$

$$y_{k+1} = 2/3\, y_k + \cos(4\pi x_k)$$

$$z_{k+1} = 2/3\, z_k + \sin(4\pi x_k)$$

Figure 4
Attractor Cross Section at $x = 1/2$



$$x_{k+1} = 3x_k \mod 1$$

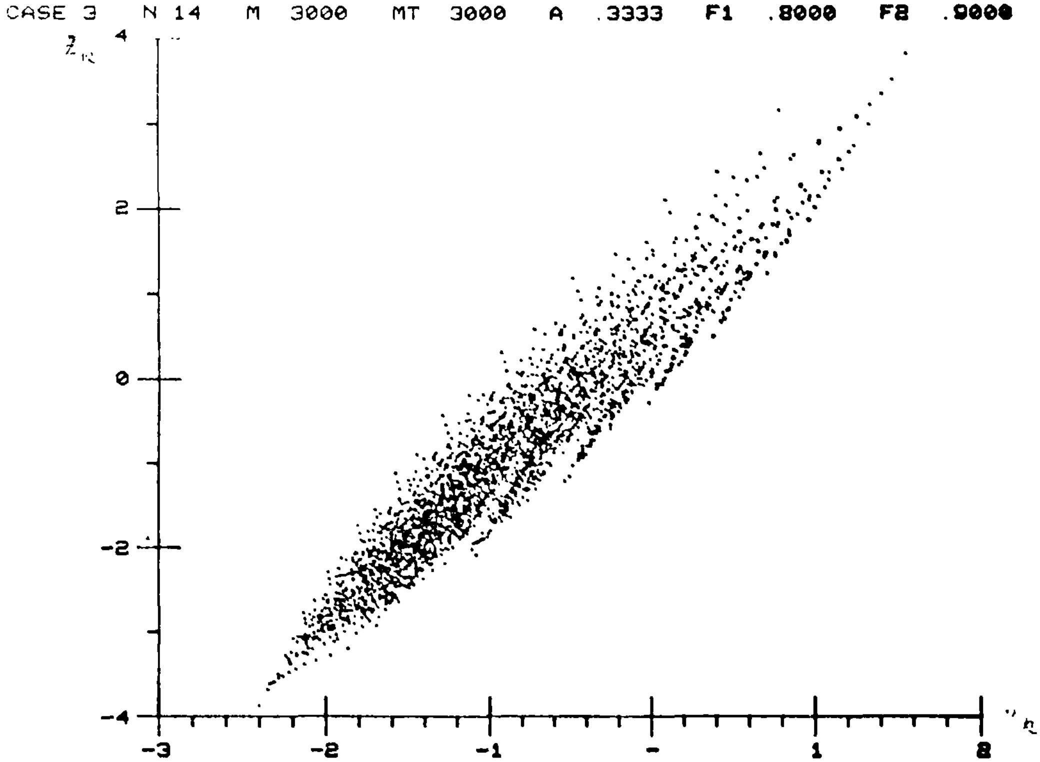
$$y_{k+1} = 2/3\ y_k + \cos(4\pi x_k)$$

$$z_{k+1} = 2/3\ z_k + \sin(4\pi x_k)$$

Note: The expansion coefficient here is 3.0 and Det $f_3 = 4/3$

14600 points plotted

Figure 5



$$x_k = 2x_{k-1} \mod 1$$
$$y_k = 0.8y_{k-1} + \cos 4\pi x_{k-1}$$
$$z_k = 0.9z_{k-1} + \cos 4\pi x_{k-1}$$

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