CALCULATING BIFURCATION INVARIANTS AS ELEMENTS IN THE HOMOTOPY OF THE GENERAL LINEAR GROUP

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In [2], a result was established that guarantees the zeroes of a continuous parameterized function bifurcate if a certain invariant is not zero. This invariant \( \gamma_j \) is an element of a quotient group of a homotopy group of the infinite general linear group \( GL \). In applications [3, 7, 9, 10], a considerable amount of effort is expended computing this invariant. The purpose of this note is to present two methods for determining \( \gamma_j \). One method reduces the problem to a question in linear algebra that in a large number of cases makes the calculation a triviality. The other, which can yield results when the first method cannot, reduces the problem to evaluating a certain integral.

We recall the definition of the invariant. Let \( B \) be a Banach space (possibly finite-dimensional). Let \( \mathcal{U} \) be an open set in \( \mathbb{R}^n \times B \) containing the origin \((0, 0)\), and let \( f: \mathcal{U} \rightarrow B \) be a continuous map. Usually the \( \mathbb{R}^n \) variable, denoted \( \lambda \), is considered a parameter. Suppose that \( f(\lambda, 0) = 0 \) for \( \lambda \) in some neighborhood \( N \) of the origin in \( \mathbb{R}^n \), and that on this neighborhood the derivative \( L(\lambda) = D_\lambda f(\lambda, u) \mid_{u=0}: N \rightarrow \text{Hom}(B, B) \) exists and is continuous on \( N \). Here \( \text{Hom}(E, B) \) denotes the bounded linear maps of \( B \) to \( B \). Assume that \( L(\lambda) \) is invertible in \( \text{Hom}(B, B) \) for \( \lambda \neq 0 \). If \( B \) is infinite-dimensional, some (usually standard) restriction must be put on the form of \( f \). For example, in [2] it is assumed \( f \) is of the form (identity + compact). The present results work if \( L(\lambda) \) is Fredholm; for convenience we henceforth assume \( B = \mathbb{R}^M \). In that case, the space of invertible maps in \( \text{Hom}(B, B) \) is \( GL(M) \), the general linear group. Let \( S^{n-1} \) be a small sphere around 0 in \( \mathbb{R}^n \). Mapping \( S^{n-1} \) by \( L \) yields a homotopy element \( \gamma \in \pi_{n-1}(GL(M)) \). Stabilize \( \gamma \) to an element \( \gamma_j \in \pi_{n-1}(GL) \). The structure of \( \pi_{n}(GL) \) is determined by the Bott Periodicity Theorem and can be read from the following table (where \( Z \) denotes the group of integers and \( \mathbb{Z}/2\mathbb{Z} \) is the group of two elements).

\[
\begin{array}{c|cccccccc}
\mod 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\pi_{n-1}(GL) & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}
\end{array}
\]

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The result of [2] states that if \( y_i \neq 0 \) (if \( n = 1 \) or \( 2 \mod 8 \)) or if \( y_i \) is not divisible by some number \( b_k \) (if \( n = 4k \)), then there is a connected family of zeroes of \( f \) bifurcating from the origin \((0, 0)\). It is easy to check that \( b_k \) is always non-zero and even (in fact divisible by 8).

**Results**

We first give a result that determines the value of \( y_i \) (if \( n = 1 \) or \( 2 \mod 8 \)) or its value modulo 2 (if \( n = 4k \)).

Assume that \( L(\lambda) \) has a first order Taylor expansion around 0. So

\[
L(\lambda) = A_o + A_1 \lambda + O(|\lambda|^2).
\]

We want to assure that the behavior of \( f \) is determined by the low-order terms, so we assume

(*) there exists \( \varepsilon > 0 \) so that \(|L(\lambda) \cdot u| \geq \varepsilon |\lambda| |u|\) for \( \lambda \) near zero.

Let \( m \) denote the dimension of the kernel of \( A_o \). Let \( c_n \) be the integer defined as follows:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
c_n & 1 & 2 & 4 & 4 & 8 & 8 & 8 & 8 \\
\end{array}
\]

\( c_n + 8 = 16c_n \)

\( n \mid 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \)

(1) Given (*) then \( c_n \) divides \( m \).

Let \( n = 1, 2, 4, \) or \( 8 \mod 8 \). Assume (*).

(2) If \( m = \mu c_n \) where \( \mu = \mu(A_o) \) is an integer, then \( y_i \) is zero or non-zero modulo \( 2^{n-1}(G) \) according to whether \( \mu \) is even or odd. Thus if \( \mu \) is odd, bifurcation is guaranteed.

Now consider the possibilities where the parameter and variable spaces can have the same dimension.

(3) If \( M = n \) and (*) holds, then \( n = 1, 2, 4, \) or \( 8 \). Moreover in these cases, bifurcation is guaranteed.

These cases are important because they are precisely the ones where classic implicit-function techniques can be used to locate the bifurcating set. Hence, in these cases, the bifurcating set has dimension \( n \), which is not true in general. Result (3) was first observed by the second-named author, who also realized it was related to J.F. Adams' results about vector fields on spheres. This generalized to result (1) (see [4]) and then the first-named author proved result (2).

If \( n = 1 \), we have precisely the case considered by Rabinowitz of nonlinear real
In this case the parity of the multiplicity of the eigenvalue determines \( \gamma^f \). In particular, if the eigenvalue is simple, bifurcation occurs. Condition (*) is sometimes expressed by saying the eigenvalues go through 0 with non-zero derivative. Using \( n = 2 \), Izé considered complex eigenvalue problems [7]. Again the parity of the multiplicity of the eigenvalue determines \( \gamma^f \). Another use of the case \( n = 2 \) is to prove Hopf bifurcation. We show the present result is applicable for this result also. We assume the reader is familiar with the set-up of [3], and merely isolate the necessary algebra.

Let \( \mu \mapsto F(\mu) \in \text{Hom}(\mathbb{R}^M, \mathbb{R}^M) \) be a \( C^1 \) map where \( \mu \) lives on some real interval around 0. Suppose \( F(0) \) is not singular and that it has a pair of conjugate purely imaginary eigenvalues \( \pm i \beta \). Let Mult (\( i \beta \)) be the set \{ \( ik_1 \beta, ik_2 \beta, \ldots, ik_r \beta \} \) of eigenvalues of \( F(0) \) which are positive integral multiples of \( i \beta \) (counted with multiplicities). For \( \mu \) near 0, there is a set Mult\(_n\) (\( i \beta \)) of exactly \( r \) eigenvalues of \( F(\mu) \) which are near the elements of Mult (\( i \beta \)). Let \( \alpha = \alpha(\mu) \) be the minimum of the real parts of the elements of Mult\(_n\) (\( i \beta \)). Suppose \( \alpha \) satisfies

\[
(*) \quad \text{there exists } \varepsilon > 0 \text{ such that } |\alpha(\mu)| \geq \varepsilon |\mu|.
\]

Thus the eigenvalues of Mult\(_n\) (\( i \beta \)) go through the imaginary axis with non-zero derivative. Let \( t \) be a real variable and \( t_0 = 2\pi \beta^{-1} \).

(4) Given (**), then the function

\[
L(\mu, t) = \exp(i\mu) - I \in \text{GL}(M)
\]

has a first-order Taylor expansion around \((\mu, t) = (0, t_0)\) and satisfies (*). Moreover, the dimension of the kernel of \( L(0, t_0) \) is \( 2r \).

Thus in applications, if \( r \) is odd, Hopf bifurcation occurs.

Clearly the fact that only the parity of \( \gamma^f \) is determined if \( n = 4k \) limits the applicability of result (2). It will be seen in the course of the proof that a more precise result is not possible if the data is the dimension of the kernel of \( A_0 \). We now give a result also due to the first-named author that completely determines \( \gamma^f \) if \( n = 4k \). This result does not require assumption (*) or even that \( L \) has a Taylor expansion. It is required that \( L \) be \( C^2 \) near \( S^{4k-1} \) (any \( L \) can be so approximated). Consider the following formal procedure. Let \( t \) be a real variable and consider the matrix of differential 1-forms \( w = r dL \cdot L^{-1} \) on \( \mathbb{R} \times S^{4k-1} \) (i.e., \( w_{ij} = \sum_k r dL_{ik} (L^{-1})_{kj} \)). Then \( dw \) has components in the commutative algebra of even-dimensional forms on \( \mathbb{R} \times S^{4k-1} \), so determinant \((1 + dw)\) is well-defined. The coefficient \( w_{4k-1} \) of \( t^{4k-1} dt \) is a \( 4k - 1 \) form on \( S^{4k-1} \). Let \( S^{4k-1} \) have radius \( R \). Let

\[
K_k = \begin{cases} 
\frac{(2k-1)!}{(2\pi)^{2k} (4k-1)!}, & k \text{ even}, \\
\frac{(2k-1)!}{(2\pi)^{2k} 2(4k-1)!}, & k \text{ odd}.
\end{cases}
\]
(5) The number \( \kappa = (K_k/R^{2k-1}) \int_{S^{2k-1}} w_{2k-1} \) is an integer and \( \gamma_i^* \) is \( \kappa \) times a generator in \( \pi_{2k-1}(GL) = \mathbb{Z} \).

Proofs

Proof of Result (1). First note that since
\[
| (L(\lambda) - (A_0 + A_1\lambda)) \cdot u | \leq \epsilon_1 | \lambda \cdot u |
\]
for some \( \epsilon_1 > 0 \), condition (*) must also hold for the function \( (A_0 + A_1\lambda) \cdot u \). Thus \( A_0 + A_1\lambda \) is non-singular if \( \lambda \neq 0 \). Moreover, also by (*), the homotopy
\[
(1 - \tau)L(\lambda) + \tau (A_0 + A_1\lambda) = A_0 + A_1\lambda + (1 - \tau)0(\lambda \cdot u)
\]
between \( L(\lambda) \) and \( A_0 + A_1\lambda \) is a homotopy in \( \text{CL}(M) \) if \( \lambda \neq 0 \). Thus we may restrict our attention to \( A_0 + A_1\lambda \).

Let \( K \) be the kernel of \( A_0 \), and \( K' \) the cokernel. Both are copies of \( \mathbb{R}^m \) for some \( m \) (this is also true in the infinite-dimensional case, since \( L(\lambda) \) is Fredholm of index zero). Write \( B = K \oplus K_1 = K' \oplus \text{image} \ A_0 \). Then \( A_0 \mid R^n \times K_1 : K_1 \to \text{image} \ A_0 \) is an isomorphism and \( (A_0 + A_1\lambda) \mid S^{n-1} \times K_1 \) can be homotoped to \( A_0 \mid \{0\} \times K_1 \). The stable homotopy class of \( (A_0 + A_1\lambda) \mid R^n \times B : R^n \times B \to B \) is the same as that of \( A_1 \mid R^n \times K : R^n \times K \to K' \) (which we continue to denote \( A_1 \)); we may thus assume \( B = R^n \) and \( A_0 = 0 \). A bilinear map \( A_1 : R^n \times R^n \to R^n \) which satisfies \( A_1(\lambda, u) \neq 0 \) for \( |\lambda| \cdot |u| \neq 0 \) we call a non-singular action of \( R^n \) on \( R^n \). We call \( m \) the dimension of the action.

Let \( S^{n-1} \) be the unit sphere in \( R^n \). Recall that the group \( Z/2Z \) acts on \( S^{n-1} \) by the antipodal action and that the quotient space is projective space \( RP^{n-1} \). The group \( Z/2Z \) also operates on \( S^{n-1} \times R \) by the antipodal map on each factor and the quotient space is a fiber bundle over \( RP^{n-1} \) (the projection being induced by the projection \( S^{n-1} \times R \to S^{n-1} \)). This bundle and its total space are denoted \( \xi_{n-1} \) and called the canonical bundle over \( RP^{n-1} \). If we let \( Z/2Z \) operate by the antipodal map on each factor of \( S^{n-1} \times R^n \), the quotient is the \( p \)-fold Whitney sum bundle \( p\xi_{n-1} \) over \( RP^{n-1} \). Let \( KO(RP^{n-1}) \) denote the group of equivalence classes of stable bundles over \( RP^{n-1} \). A stable bundle is a bundle \( \eta \) together with all bundles of the form \( \eta + \) (trivial bundle) and two stable bundles are equivalent if any of their respective representatives are equivalent. In particular, a trivial bundle over \( X \) is 0 in \( KO(X) \). The group \( KO(RP^{n-1}) \) was computed by J.F. Adams [1]; it is cyclic of order \( c_n \) and a generator is the class of \( \xi_{n-1} \).

The action \( A_1 \) restricts to a map \( A_1 : S^{n-1} \times R^n \) which factors through \( A \mid m\xi_{n-1} \to R^n \) that is an isomorphism on each fiber and so is a trivialization of \( m\xi_{n-1} \). Thus \( m\xi_{n-1} \) is 0 in \( KO(RP^{n-1}) \) and hence \( c_n \) divides \( m \). This proves Result (1). (Result 1 was proved by a cruder technique in [4].)

Proof of Result (2). Consider the set of equivalence classes of non-singular actions \( A_1 : R^n \times R^n \to R^n \) for fixed \( n \). Two such actions on \( R^n \) and \( R^m \) induce one on
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\[ R^{n+m} \]; the equivalence classes thus form a semi-group. Let \( \mathcal{A}_n \) denote the associated Grothendieck group. For \( q > n \), we fix \( R^n \subset R^q \). Clearly there is induced a restriction map \( \text{res}: \mathcal{A}_q \to \mathcal{A}_n \). An element of \( \widetilde{KO}(\mathbb{RP}^{n+1}/\mathbb{RP}^n) \) may be considered a bundle over \( \mathbb{RP}^n \) with an explicit trivialization over \( \mathbb{RP}^{n+1} \) \([5, \text{chap. 9}]\). Thus we may define a group homomorphism

\[ \beta : \mathcal{A}_n \to \widetilde{KO}(\mathbb{RP}^{n+1}/\mathbb{RP}^n), \]

by mapping \( A_1 : R^n \times R^m \to R^m \) to the class of \( m \xi - 1 \) with the trivialization over \( \mathbb{RP}^n \) effected by \( A_1 \). Clearly if \( A_1 \in \text{res} \mathcal{A}_q \) then \( \beta(A_1) = 0 \). Thus \( \beta \) induces a map

\[ \beta : \mathcal{A}_n/\text{res} \mathcal{A}_q \to \widetilde{KO}(\mathbb{RP}^{n+1}/\mathbb{RP}^n). \]

Consider now actions \( A_1 : R^n \times R^m \to R^n \) where \( |A_1(\lambda, u)| = |\lambda| \cdot |u| \). Such are called orthogonal actions. Let \( \mathcal{A}_n^0 \) denote the resultant Grothendieck group of equivalence classes. There is the inclusion \( i : \mathcal{A}_n^0 \to \mathcal{A}_n \) and

\[ \beta^0 = \beta i : \mathcal{A}_n^0 \to \widetilde{KO}(\mathbb{RP}^{n+1}/\mathbb{RP}^n). \]

All the above is quite standard in the orthogonal case \([5, \S 15], [6, \S 15.11]\), the structures of \( \mathcal{A}_n^0 \) and \( \text{res} : \mathcal{A}_n^0+1 \to \mathcal{A}_n^0 \) are completely known \([5, \S 5], [6, \text{Chap. 11}]\), and in particular, the Bott Periodicity Theorem implies that for \( q = n + 1 \),

\[ \beta^0 : \mathcal{A}_n^0/\text{res} \mathcal{A}_n^0 \to \widetilde{KO}(\mathbb{RP}^{n+1}/\mathbb{RP}^n) = \widetilde{KO}(S^n) = \pi_n(BO) = \pi_{n-1}(GL) \]

is an isomorphism (see \([5, \text{Th. 11.5, \S 15}]\)). If \( n = 1 \) or 2 modulo 8, the group \( \mathcal{A}_n^0 \) is infinite cyclic and \( \text{res} \mathcal{A}_n^0+1 = 2\mathcal{A}_n^0 \). Thus the dimension of an orthogonal action \( A_1 \) completely determines \( \beta^0(A_1) \). The same is true for a non-singular \( A_1 \). If \( n \) is a multiple of 4, the group \( \mathcal{A}_n^0 \) is free abelian on two generators, both of dimension \( c_n \). Unfortunately, \( \text{res} \mathcal{A}_n^0+1 \) is generated by the sum of the two generators. Thus the dimension of \( A_1 \) does not determine \( \beta^0(A_1) \), but only its value modulo \( 2\pi_{n-1}(GL) \). Hence result 2 must be stated as is.

To complete the proof of (2), we must show that if \( L(\lambda) = A_1 \lambda \), then \( \gamma_i = \beta(A_1) \).

To do this we consider the following standard description of a bundle over a sphere. Let \( D^n \) be an \( n \)-dimensional disk with boundary \( S^{n-1} \). Let \( g : S^{n-1} \to GL(r) \) be a continuous map. Consider \( D^n \times R' \) modulo the identification \( (s, g(s)) = (s', g(s')) \) for \( s, s' \) on the boundary \( S^{n-1} \). This then gives an \( r \)-dimensional bundle over \( D^n \) with an explicit trivialization over \( S^{n-1} \). This process stabilizes and thus we have an element of \( \widetilde{KO}(D^n/S^{n-1}) = \widetilde{KO}(S^n) \). The equivalence class depends only on the stable homotopy class of \( g \), so we receive a map \( \pi_{n-1}(GL) \to \widetilde{KO}(S^n) \). This map expresses the isomorphism between \( \widetilde{KO}(S^n) \) and \( \pi_{n-1}(GL) \) (it is easy to see that a map \( g \) exists for any bundle). Let \( D^n \) be the space of points \( (\lambda_1, \ldots, \lambda_n) \) with \( \Sigma \lambda_i = 1 \). Let \( \mathbb{RP}^n \) be the space of points \([y_0, \ldots, y_n]\) where \( \Sigma y_i = 1 \) and \([y_0, \ldots, y_n] = [y_0, \ldots, -y_n] \). We consider \( \mathbb{RP}^{n-1} \) as the subspace with \( y_0 = 0 \). Define \( t : D^n \to \mathbb{RP}^n \) by

\[ t(\lambda_1, \ldots, \lambda_n) = \left[1 - \Sigma \lambda_i, \lambda_1, \ldots, \lambda_n \right]. \]
Then \( t(S^{n-1}) = \mathbb{R}P^{n-1} \) and \( t \) induces a homeomorphism

\[
\tilde{t} : \mathbb{R}P^n/\mathbb{R}P^{n-1} \to D^*/S^{n-1}
\]

of \( S^n \) to itself.

We now specialize to the case we have been considering of \( m\xi_n \) over \( \mathbb{R}P^n \) trivialized over \( \mathbb{R}P^{n-1} \) by \( A_1 \). We consider \( t^*m\xi_n \) and find it is canonically a product \( D^n \times \mathbb{R}^m \) over \( D^n \). The trivialization \( A_1 \) corresponds to the map \( g : S^{n-1} \to GL(m) \) given by \( g(s) = A_1 \lambda \) if \( s = (\lambda_1, \ldots, \lambda_n) \in S^{n-1} \). This is precisely what is needed to show \( \gamma_1 = \beta(A_1) \). Thus Result (2) is proved.

**Proof of Result (3).** Clearly Result (3) is a corollary of (1) and (2). We turn to (4).

**Proof of Result (4).** If the eigenvalues of a matrix \( A(\mu) \) are \( \nu_1(\mu), \ldots, \nu_\mu(\mu) \), then the eigenvalues of \( \exp(tA(\mu)) - I \) are \( e^{\nu_1(\mu)} - 1, \ldots, e^{\nu_\mu(\mu)} - 1 \). If there exists \( \epsilon > 0 \) such that for each \( \nu(\mu) \)

\[
(\ast\ast\ast) \quad |e^{\nu(\mu)} - 1| \geq \epsilon (\mu^2 + (t - t_0)^2),
\]

then (\ast) is true for \( \exp(tA(\mu)) - I \). If \( \nu(\mu) \notin \text{Mult}(i\beta) \), then (\ast\ast\ast) is trivial. We leave it to the reader to do the elementary estimates to show that (\ast\ast\ast) is also true for \( \nu(\mu) \notin \text{Mult}(i\beta) \) in light of (\ast\ast), and thus complete the proof of (4).

**Proof of Result (5).** Result (5) is proved by evaluating a characteristic class by curvature forms. Appendix C of [8] is a general reference. Recall a connection on a bundle \( \xi \) over a manifold \( M \) is an \( \mathbb{R} \)-linear map \( \nabla \) from sections of \( \xi \) to sections of \( \tau^* \otimes \xi \) (where \( \tau^* \) is the dual of the tangent bundle, i.e., 1-forms) satisfying

\[
\nabla(fs) = df \otimes s + f\nabla s
\]

for \( f \) a function on \( M \) and \( s \) a section. Let \( U_n \) be a coordinate neighborhood of \( \xi \) so that \( \xi \mid U_n = U_n \times \mathbb{R}^p \) and let \( s_1, \ldots, s_p \) be the standard basis cross sections over \( U_n \). Over \( U_n \), we can write

\[
\nabla s_i = \sum_j \omega_s^i \otimes s_j,
\]

where \( \omega_s \) is a matrix of 1-forms. Let \( g = g_{ab} : U_a \cap U_b \to GL(p) \) be the gluing maps of \( \xi \), [6, Chap. 5], that is \( (m, x)_a = (m, g(x))_b \) for \( (m, x)_a \in (U_a \cap U_b) \times \mathbb{R}^p \subset U_a \times \mathbb{R}^p \) and \( (m, g(x))_b \in (U_a \cap U_b) \times \mathbb{R}^p \subset U_b \times \mathbb{R}^p \). It is routine to establish the following compatibility formula for the components of \( \nabla \):

\[
dg = \omega_s^a \cdot g - g \cdot \omega_s^a
\]

\[
\left(\text{i.e.,} \quad \nabla g = \sum_k (\omega_s^{a_k} g_{a_k} - g_{a_k} \cdot \omega_s^{a_k})\right).
\]

We specialize to the case of \( M = S^n = D^n \cup D^n \) where \( D^n \) is the union of \( S^{n-1} \), the sphere of radius \( R \) in \( \mathbb{R}^n \), together with its interior in \( \mathbb{R}^n \), and \( D^n \) is another disk
such that $D^* \cap D^\circ$ is an annulus $A$ in $\mathbb{R}^n$ of radii $\frac{1}{2}R$ and $R$. Let $U_a$ be the interior of $D^*$ and $U_\mu$ be the interior of $D^\circ$. Let $g : A \to \text{GL}(p)$ be $g(m) = L(Rm/|m|)$ where $L : S^{n-1} \to \text{GL}(p)$. Note that $g$ is radially invariant. We can require $\omega^\beta = 0$ on $D_\circ$; hence on $A$ the forms $\omega^\beta$ must satisfy $dg = \omega^\alpha \cdot g$ or $\omega^\alpha = dg \cdot g^{-1}$. Let $\varphi : [0, R] \to [0, 1]$ be a smooth monotone function which is 0 near 0 and 1 on $(\frac{1}{2}R, R)$. Let $r = |m|$ be the radial component in $U_a$. Define

$$
\omega(m) = \varphi(r) \frac{1}{L^{-1}} \frac{dL}{r}.
$$

Then $\omega = \omega^\alpha$ is a connection form on $U_a$ (= interior $D^*$) compatible with the flat form $\omega^\alpha = 0$ on $D^\circ$. The bundle $\xi$ has gluing map or characteristic map $L : S^{n-1} \to \text{GL}(p)$. Note since $L \cdot L^{-1} = I$, that

$$
dL \cdot L^{-1} + L \cdot dL^{-1} = 0.
$$

Thus

$$
\omega = \varphi \frac{dL \cdot L^{-1}}{L} = - \varphi \frac{L \cdot dL^{-1}}{L}.
$$

The curvature form (a 2-form) is

$$
\Omega = d\omega - \omega \wedge \omega.
$$

In $\beta$ coordinates $\Omega = 0$, and in $\alpha$ coordinates

$$
\Omega = d\varphi \wedge dL \cdot L^{-1} - \varphi \frac{dL \cdot L^{-1}}{L} + \varphi dL \cdot L^{-1} \wedge \varphi L \cdot dL^{-1}
$$

$$
= d\varphi \wedge dL \cdot L^{-1} - (\varphi - \varphi^2) dL \wedge dL^{-1}.
$$

Let $n = 4k$. Let $p_k$ denote the $k$th Pontrjagin class of the bundle $\xi$. The Pontrjagin number $\langle p_k, [S^{4k}] \rangle$ determines the class of $\gamma_f$ in $\pi_{4k-1}(\text{GL}) = \mathbb{Z}$. In fact

$$
\gamma_f = \frac{1}{\alpha (2k - 1)!} \langle p_k, [S^{4k}] \rangle \cdot \text{generator}
$$

where $\alpha = 1$ or 2 according to whether $k$ is even or odd. On the other hand, $\langle p_k, [S^{4k}] \rangle$ can be calculated from $\Omega$. One considers the expression

$$
\text{determinant } (1 + \tau \Omega) = 1 + \tau \sigma_1(\Omega) + \tau^2 \sigma_2(\Omega) + \cdots
$$

where $\sigma_i(\Omega)$ is a 2i-form. Then

$$
\langle p_k, [S^{4k}] \rangle = \frac{1}{(2\pi)^k} \int_{S^{4k}} \sigma_{2k}(\Omega).
$$

The prescription for $w_{4k-1}$ is precisely such that

$$
\sigma_k(\Omega) = \begin{cases} 
-(\varphi - \varphi^2)^{2k-1} d\varphi \wedge w_{4k-1}, & \text{in } \alpha \text{ coordinates}, \\
0, & \text{in } \beta \text{ coordinates}.
\end{cases}
$$

Therefore

$$
\int_{S^{4k}} \sigma_k(\Omega) = \int_{D^\circ} -(\varphi - \varphi^2)^{2k-1} d\varphi \wedge w_{4k-1}.
$$
\[ = \int_{s^{4k-1}} w_{4k-1} \int_0^1 (\varphi - \varphi'^2)^{4k-1} d\varphi \]
\[ = -\frac{(2k-1)^2}{(4k-1)!} \int_{s^{4k-1}} w_{4k-1}. \]

Collecting the various constants, we get Result (5).

We finish with a problem in linear algebra. We had to be somewhat careful to distinguish between orthogonal actions \( A : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m \) and the more general non-singular actions. Orthogonal actions have been completely classified via the theory of Clifford algebras. Usually, when working up to homotopy, it is not necessary to differentiate between non-singular and orthogonal objects. Such is the case for example in the theory of bundles. Here the situation is not at all clear, so we phrase the following two questions:

1. Can every non-singular action of \( \mathbb{R}^n \) on \( \mathbb{R}^m \) be deformed through non-singular actions to an orthogonal action? ("Yes" for \( n = 1 \) or 2.)
2. Is \( i : \mathfrak{a}_n^U \rightarrow \mathfrak{a}_n \) an isomorphism?

References