



---

On the Existence of Invariant Measures for Piecewise Monotonic Transformations

Author(s): A. Lasota and James A. Yorke

Source: *Transactions of the American Mathematical Society*, Vol. 186 (Dec., 1973), pp. 481-488

Published by: [American Mathematical Society](#)

Stable URL: <http://www.jstor.org/stable/1996575>

Accessed: 18/08/2013 17:02

---

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



American Mathematical Society is collaborating with JSTOR to digitize, preserve and extend access to *Transactions of the American Mathematical Society*.

<http://www.jstor.org>

## ON THE EXISTENCE OF INVARIANT MEASURES FOR PIECEWISE MONOTONIC TRANSFORMATIONS(1)

BY

A. LASOTA AND JAMES A. YORKE

**ABSTRACT.** A class of piecewise continuous, piecewise  $C^1$  transformations on the interval  $[0, 1]$  is shown to have absolutely continuous invariant measures.

**1. Introduction.** The purpose of this note is to prove the existence of absolutely continuous invariant measures for a class of point-transformations of the unit interval  $[0, 1]$  into itself. Our main result is Theorem 1 which generalizes some previous results of A. Rényi [5], A. O. Gel'fond [2], W. Parry [4] and A. Lasota [3]. It gives, also, a positive answer to a conjecture of S. Ulam [7, p. 74]. Theorem 1 is stated for a piecewise monotonic function with a finite number of discontinuities but it can be easily extended to some piecewise monotonic functions with infinite number of discontinuities.

Our method is different from the methods of the above mentioned authors. Firstly we explore the fact that the Frobenius-Perron operator corresponding to the point-transformation under consideration has the property of sometimes shrinking the variation of the function. Secondly to prove the existence of invariant measures we use the abstract ergodic theorem which enables us to make our proofs constructive. The advantage of this method is that we do not require that our mappings be local homeomorphisms nor that they generate an exact endomorphism in the sense of Rohlin [6], a property that has been the typical requirement for previous work. §4 describes some extensions, including an extension to higher dimensions.

**2. Existence theorem.** Denote by  $(L_1, \|\cdot\|)$  the space of all integrable functions defined on the interval  $[0, 1]$ . Lebesgue measure on  $[0, 1]$  will be denoted by  $m$ . Let  $\tau: [0, 1] \rightarrow [0, 1]$  be a measurable nonsingular transformation.

---

Received by the editors November 9, 1972 and, in revised form, February 13, 1973.

*AMS (MOS) subject classifications* (1970). Primary 28A65.

*Key words and phrases.* Frobenius-Perron operator, invariant measures.

(1) The research of both authors was partially supported by the National Science Foundation under grant GP-31386x.

Copyright © 1974, American Mathematical Society

“Nonsingularity” means that  $m(\tau^{-1}(A)) = 0$  whenever  $m(A) = 0$  for a measurable set  $A$ . Given  $\tau$  we define the Frobenius-Perron operator  $P_\tau: L_1 \rightarrow L_1$  by the formula

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}([0, x])} f(s) ds.$$

It is well known that the operator  $P_\tau$  is linear and continuous and satisfies the following conditions:

- (a)  $P_\tau$  is positive:  $f \geq 0 \Rightarrow P_\tau f \geq 0$ ;
- (b)  $P_\tau$  preserves integrals

$$\int_0^1 P_\tau f dm = \int_0^1 f dm, \quad f \in L_1;$$

(c)  $P_\tau n = P_\tau^n$  ( $\tau^n$  denotes the  $n$ th iterate of  $\tau$ );

(d)  $P_\tau f = f$  if and only if the measure  $d\mu = f dm$  is invariant under  $\tau$ , that is  $\mu(\tau^{-1}(A)) = \mu(A)$  for each measurable  $A$ .

A transformation  $\tau: [0, 1] \rightarrow R$  will be called *piecewise  $C^2$* , if there exists a partition  $0 = a_0 < a_1 < \dots < a_p = 1$  of the unit interval such that for each integer  $i$  ( $i = 1, \dots, p$ ) the restriction  $\tau_i$  of  $\tau$  to the open interval  $(a_{i-1}, a_i)$  is a  $C^2$  function which can be extended to the closed interval  $[a_{i-1}, a_i]$  as a  $C^2$  function.  $\tau$  need not be continuous at the points  $a_i$ .

**Theorem 1.** *Let  $\tau: [0, 1] \rightarrow [0, 1]$  be a piecewise  $C^2$  function such that  $\inf |\tau'| > 1$ . Then for any  $f \in L_1$  the sequence*

$$\frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f$$

*is convergent in norm to a function  $f^* \in L_1$ . The limit function has the following properties:*

- (1)  $f \geq 0 \Rightarrow f^* \geq 0$ .
- (2)  $\int_0^1 f^* dm = \int_0^1 f dm$ .
- (3)  $P_\tau f^* = f^*$  and consequently the measure  $d\mu^* = f^* dm$  is invariant under  $\tau$ .
- (4) The function  $f^*$  is of bounded variation; moreover, there exists a constant  $c$  independent of the choice of initial  $f$  such that the variation of the limiting  $f^*$  satisfies the inequality(2)

$$\bigvee_0^1 f^* \leq c \|f\|.$$

---

(2) Here and in what follows the symbol  $\bigvee_a^b f$  as well as  $\bigvee_{[a, b]} f$  denote the variation of  $f$  over the closed interval  $[a, b]$ .

We point out in Theorem 3 that it is sufficient to assume just that some iterate of  $\tau$  satisfy the derivative condition.

**Proof.** Write  $s = \inf|r'|$  and choose a number  $N$  such that  $s^N > 2$ . It is easy to see that the function  $\phi = r^N$  is piecewise  $C^2$ . Denote by  $b_0, \dots, b_q$  the corresponding partition for  $\phi$ . Writing  $\phi_i$  for the corresponding  $C^2$  functions we have

$$(5) \quad |\phi'_i(x)| \geq s^N, \quad x \in [b_{i-1}, b_i], \quad i = 1, \dots, q,$$

Computing the Frobenius-Perron operator for  $\phi$  we obtain

$$(6) \quad P_\phi f(x) = \sum_{i=1}^q f(\psi_i(x))\sigma_i(x)\chi_i(x)$$

where  $\psi_i = \phi_i^{-1}$ ,  $\sigma_i(x) = |\psi'_i(x)|$  and  $\chi_i$  is the characteristic function of the interval  $J_i = \phi_i([b_{i-1}, b_i])$ . From (5) it follows that

$$(7) \quad |\sigma_i(x)| \leq s^{-N}, \quad x \in J_i, \quad i = 1, \dots, q.$$

By its very definition the operator  $P_\phi$  is defined as a mapping from  $L_1$  into  $L_1$  but the formula (6) enables us to consider  $P_\phi$  as a map from the space of functions defined on  $[0, 1]$  into itself.

Let  $f$  be a given function of bounded variation over  $[0, 1]$ . From (6) and (7) it follows that

$$(8) \quad \bigvee_0^1 P_\phi f \leq \sum_{i=1}^q \bigvee_{J_i} (f \circ \psi_i)\sigma_i + s^{-N} \sum_{i=1}^q (|f(b_{i-1})| + |f(b_i)|).$$

In order to evaluate the first sum we write

$$\begin{aligned} \bigvee_{J_i} (f \circ \psi_i)\sigma_i &= \int_{J_i} |d((f \circ \psi_i)\sigma_i)| \\ &\leq \int_{J_i} |f \circ \psi_i| |\sigma'_i| dm + \int_{J_i} \sigma_i |d(f \circ \psi_i)| \\ &\leq K \int_{J_i} |f \circ \psi_i| \sigma_i dm + s^{-N} \int_{J_i} |d(f \circ \psi_i)| \end{aligned}$$

where  $K = \max|\sigma'_i|/\min(\sigma_i)$ . Changing the variables we obtain

$$(9) \quad \bigvee_{J_i} (f \circ \psi_i)\sigma_i \leq K \int_{b_{i-1}}^{b_i} |f| dm + s^{-N} \int_{b_{i-1}}^{b_i} |df|.$$

In order to evaluate the second term in (8) we write

$$(10) \quad |f(b_{i-1})| + |f(b_i)| \leq \bigvee_{b_{i-1}}^{b_i} f + 2d_i$$

where  $d_i = \inf \{|f(x)| : x \in [b_{i-1}, b_i]\}$ . On the other hand we have an obvious inequality

$$(11) \quad d_i \leq b^{-1} \int_{b_{i-1}}^{b_i} |f| \, dm$$

where  $b = \min_i (b_i - b_{i-1})$ . From (10), (11) it follows that

$$(12) \quad \sum_{i=1}^q (|f(b_{i-1})| + |f(b_i)|) \leq \frac{1}{b} \|f\| + 2b^{-1} \|f\|.$$

Applying (12) and (9) to (8) we obtain  $\int_0^1 P_\phi f \leq \alpha \|f\| + \beta \int_0^1 f$  where  $\alpha = (K+2b^{-1})$  and  $\beta = 2s^{-N} < 1$ .

Now, for the same function  $f$ , let us write  $f_k = P_\tau^k f$ . Since  $P_\tau^N = P_\phi$  we have

$$\int_0^1 f_{Nk} \leq \alpha \|f_{N(k-1)}\| + \beta \int_0^1 f_{N(k-1)} \leq \alpha \|f\| + \beta \int_0^1 f_{N(k-1)}$$

and consequently

$$(13) \quad \limsup_{k \rightarrow \infty} \int_0^1 f_{Nk} \leq \alpha(1 - \beta)^{-1} \|f\|.$$

The last inequality and the condition  $\|f_k\| \leq \|f\|$  (which follows from (a) and (b)) prove that the set  $C = \{f_{Nk}\}_{k=0}^\infty$  is relatively compact in  $L_1$ . Since  $\{f_k\}_{k=0}^\infty \subset \bigcup_{k=0}^{N-1} P_\tau^k C$ , the whole sequence  $\{f_k\}_{k=0}^\infty$  is relatively compact, too. By Mazur's theorem the same is true for the sequence

$$(14) \quad \left\{ \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f \right\}.$$

The set of functions of bounded variation is dense in  $L_1$ . We have proved that for any such function  $f$  the sequence (14) is relatively compact. Therefore, we are in a position to use the Kakutani-Yosida Theorem (see [1, VIII.5.3]) which says that for any  $f \in L_1$  the sequence (14) converges strongly to a function  $f^*$  which is invariant under  $P_\tau$ . From (a) and (b) it follows that  $f^*$  satisfies (1) and (2). Therefore it remains only to prove (4). Since the operator  $P_\tau$  is given by a formula analogous to (6) it is easy to derive the inequality  $\int_0^1 P_\tau f \leq c_1 \int_0^1 f + c_2 \|f\|$  with some constants  $c_1$  and  $c_2$ . This and (13) imply the inequality

$$\limsup_{k \rightarrow \infty} \int_0^1 P_\tau^k f \leq c \|f\|$$

(with a positive constant  $c$ ) which is valid for any  $f$  with bounded variation. Consequently for any such  $f$  we have also

$$\limsup_{k \rightarrow \infty} \bigvee_0^1 \left( \frac{1}{n} \sum_{k=1}^{n-1} P_{\gamma}^k f \right) \leq c \|f\|.$$

Writing  $Q = \lim_n (1/n) \sum_{k=1}^{n-1} P_{\gamma}^k$  and using Helly's theorem we have  $\bigvee_0^1 Qf \leq c \|f\|$ , for  $f$  of bounded variation. The operator  $Q$  is linear and contractive. We may therefore apply Helly's theorem once more to extend this inequality for the closure of the set of functions of bounded variation, that is to all of  $L_1$ . This finishes the proof.

3. **A counterexample.** Now we shall show that our assumption  $\inf |\gamma'| > 1$  is essential. Consider the transformation

$$\gamma(x) = \begin{cases} x/(1-x) & \text{for } 0 \leq x < 1/2, \\ 2x-1 & \text{for } 1/2 \leq x \leq 1 \end{cases}$$

for which the assumption  $|\gamma'(x)| > 1$  is violated only at  $x = 0$ . We are going to prove that for any  $f \in L^1$  the sequence  $P_{\gamma}^n f$  converges in measure to zero. Therefore the equation  $P_{\gamma} f = f$  has only the trivial solution and there is no absolutely continuous nontrivial measure invariant under  $\gamma$ .

The proof will be given in a few steps. First we prove that for  $f_0 \equiv 1$  the sequence  $g_n(x) = x f_n(x)$ , where  $f_n = P_{\gamma}^n f_0$ , converges to a constant  $c_0$ . Then using the condition  $\|f_n\| = 1$  we derive easily that  $c_0 = 0$ , and consequently  $f_n \rightarrow 0$ . Finally by an approximation argument we may extend this result to an arbitrary sequence  $P_{\gamma}^n f$  with  $f \in L_1$ .

The Frobenius-Perron operator  $P_{\gamma}$  may be written in the form

$$P_{\gamma} f(x) = \frac{1}{(1+x)^2} f\left(\frac{x}{1+x}\right) + \frac{1}{2} f\left(\frac{1+x}{2}\right).$$

Thus for  $g_n$  we have the following recursive formula:

$$(15) \quad g_{n+1}(x) = \frac{1}{1+x} g_n\left(\frac{x}{1+x}\right) + \frac{x}{1+x} g_n\left(\frac{1+x}{2}\right), \quad g_0(x) \equiv x.$$

By an induction argument it is easy to check that  $g'_n \geq 0$  for each  $n$ . Therefore all the functions  $g_n$  are positive and increasing. According to (15) we have

$$g_{n+1}(1) = 1/2 g_n(1/2) + 1/2 g_n(1) \leq g_n(1).$$

This proves the existence of a limit  $\lim_n g_n(1) \stackrel{\text{df}}{=} c_0$ . Write  $z_0 = 1$  and  $z_{k+1} = z_k/(1+z_k)$ . According to (15) we obtain

$$g_{n+1}(z_k) = \frac{1}{1+z_k} g_n(z_{k+1}) + \frac{z_k}{1+z_k} g_n\left(\frac{1+z_k}{2}\right).$$

Fix  $k$  and suppose that  $\lim_n g_n(x) = C_0$  for  $z_k \leq x \leq 1$ . (This is certainly true for  $k = 0$ .) Since  $z_k \leq 1/2 + 1/2 z_k$ , we obtain at the limit as  $n \rightarrow \infty$

$$C_0 = \frac{1}{1 + z_k} \lim_n g_n(z_{k+1}) + \frac{z_k}{1 + z_k} C_0.$$

Thus  $\lim_n g_n(z_{k+1}) = C_0$ . Since  $g_n$  are increasing, this proves that  $\lim_n g_n(x) = C_0$  uniformly for all  $x \in [z_{k+1}, 1]$ . Therefore by an induction argument it follows that  $\lim_n g_n(x) = C_0$  in any interval  $[z_k, 1]$  and consequently, since  $\lim_k z_k = 0$ , we have  $\lim_n g_n(x) = C_0$  for all  $0 < x \leq 1$ . Hence,  $\lim_n f_n(x) = c_0/x$ . We claim that  $c_0 = 0$ . If not there would exist  $\epsilon > 0$  such that  $\int_\epsilon^1 c_0/x \, dx > 1$  and

$$\lim_n \int_\epsilon^1 f_n(x) \, dx = \int_\epsilon^1 \frac{c_0}{x} \, dx > 1$$

which is impossible since  $\|f_n\| = 1$  for each  $n$ . It can be easily proved by induction that each of the functions  $f_n$  is decreasing. Thus the convergence of  $f_n$  to zero is uniform on any interval  $[\epsilon, 1]$  with  $\epsilon > 0$ .

Now let  $f$  be an arbitrary function. We may write  $f = f^+ - f^-$  where  $f^+ = \max(0, f)$  and  $f^- = \max(0, -f)$ . Given  $\epsilon > 0$  consider a constant  $r$  such that

$$\int_0^1 (f^- - r)^+ \, dm + \int_0^1 (f^+ - r)^+ \, dm \leq \epsilon.$$

We have

$$\begin{aligned} \int_\epsilon^1 |P_\gamma^n f| \, dm &= \int_\epsilon^1 P_\gamma^n f^+ \, dm + \int_\epsilon^1 P_\gamma^n f^- \, dm \\ &\leq 2 \int_\epsilon^1 P_\gamma^n r \, dm + \int_\epsilon^1 P^n (f^+ - r) \, dm + \int_\epsilon^1 P^n (f^- - r) \, dm \\ &\leq 2r \int_\epsilon^1 P_\gamma^n 1 \, dm + \epsilon. \end{aligned}$$

Since  $P_\gamma^n 1$  converges on  $[\epsilon, 1]$  uniformly to zero we have

$$\lim_n \int_\epsilon^1 |P_\gamma^n f| \, dm = 0 \quad \text{for } \epsilon > 0$$

which proves that the sequence  $P_\gamma^n f$  converges in measure to zero.

**4. Final remarks.** Now we want to discuss some extensions of our method to other transformations. First of all we may prove an analogue of Theorem 1 for piecewise  $C^2$  transformations with a countable number of pieces.

Let  $\tau_i: \Delta_i \rightarrow [0, 1]$  be a countable sequence of  $C^2$  functions where  $\Delta_i$  is a sequence of closed intervals such that  $\sum_i m(\Delta_i) = 1$ ,  $m([0, 1] - \bigcup_i \Delta_i) = 0$ . The function  $\tau$  defined by the condition

$$\tau(x) = \tau_i(x), \quad x \in \text{interior of } \Delta_i,$$

will be called countably piecewise  $C^2$ . Note that the values of  $\tau$  on the set  $[0, 1] \setminus \bigcup_i \text{int } \Delta_i$  are arbitrary.

**Theorem 2.** Let  $\tau$  be a countably piecewise  $C^2$  function such that

$$(16) \quad \inf |\tau'(x)| > 2, \quad \sup |\tau''(x)| < \infty,$$

$$(17) \quad \tau_i(\Delta_i) = [0, 1] \quad \text{except for a finite number of intervals.}$$

Then for each  $f \in L_1$  the sequence  $(1/n) \sum_{k=0}^{n-1} P_\tau^k f$  is convergent in norm to a function  $f^*$  which satisfies conditions (1), (2), (3) and (4).

The proof of Theorem 2 is basically the same as the proof of Theorem 1. Thus it can be omitted. Let us only note that the condition (17) is essential. In fact it is easy to construct a countably piecewise linear function with the slope  $\tau' > 3$  such that  $\inf_{\epsilon \leq x \leq 1-\epsilon} \tau(x) - x$  is a positive number for each  $\epsilon$  in  $(0, 1/2)$ . (The graph of  $\tau$  lies over the diagonal.) It can be proved by elementary calculation that for any such function  $\tau$  and  $f \in L_1$ ,  $P_\tau^n f \rightarrow 0$  in measure as  $n \rightarrow \infty$ .

A close look at the proof of Theorem 1 shows that we have used only the fact that  $\sup |(r^N)'| > 2$ . Therefore, in fact, we have proved the following result.

**Theorem 3.** Let  $\tau: [0, 1] \rightarrow [0, 1]$  be a piecewise  $C^2$  function such that  $\inf |(r^{n_0})'| > 1$  for a positive integer  $n_0$ . Then for any  $f \in L_1$  the sequence  $(1/n) \sum_{k=0}^{n-1} P_\tau^k f$  is convergent in norm to a function  $f^*$  which satisfies conditions (1), (2) and (3). If, in addition,  $\inf |\tau'| > 0$  then condition (4) is also satisfied.

Observe that in our counterexample the function  $\gamma$  has the property that  $(\gamma^n)'_{x=0} = 0$  for each  $n$ . This is because the point  $(0, \gamma(0))$  lies on the diagonal.

Our techniques can be easily used to obtain new proofs of known results in higher dimensions. See [8], [9], [10] for such results. In this case  $\tau: M \rightarrow M$  is assumed  $C^1$  on a compact manifold  $M$  and the variation of a  $C^1$  function  $f$  is defined as  $\int_M |\text{grad } f(m)| dm$ . Hence in this case we do not allow discontinuities in  $\tau$ , or more generally if  $f$  is  $C^1$  on  $M \setminus \partial M$ , we must make assumptions on  $\tau$  guaranteeing  $P_\tau(f)$  is  $C^1$  on  $M \setminus \partial M$ . The techniques in [8], [9], [10] are quite different from the "bounded variation" approach of this paper.

The study of the functions  $\tau$  described arose while investigating the design of more durable high speed oil well drilling bits. The invariant measure  $f(x) dx$  describes the distribution of impacts on the surface of the bit. The durability and efficiency of the tool depends strongly on  $f$ . The first author is part of a team that has obtained patents in Poland for superior bits by slightly altering the bit shape to one with a better impact distribution  $f$ .

#### REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators. I. General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.
2. A. O. Gel'fond, *A common property of number systems*, Izv. Akad. Nauk Ser. Mat. SSSR 23 (1959), 809–814. (Russian) MR 22 #702.

3. A. Lasota, *Invariant measures and functional equations*, *Aequationes Math.* (in press).
4. W. Parry, *On the  $\beta$ -expansion of real numbers*, *Acta Math. Acad. Sci. Hungar.* 11 (1960), 401–416. MR 26 #288.
5. A. Rényi, *Representation for real numbers and their ergodic properties*, *Acta Math. Acad. Sci. Hungar.* 8 (1957), 477–493. MR 20 #3843.
6. V. A. Rohlin, *Exact endomorphisms of Lebesgue spaces*, *Izv. Akad. Nauk Ser. Mat. SSSR* 25 (1961), 499–530; English transl., *Amer. Math. Soc. Transl. (2)* 39 (1964), 1–36. MR 26 #1423.
7. S. M. Ulam, *A collection of mathematical problems*, *Interscience Tracts in Pure and Appl. Math.*, no. 8, Interscience, New York, 1960. MR 22 #10884.
8. M. S. Waterman, *Some ergodic properties of multidimensional  $F$ -expansions*, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 16 (1970), 77–103. MR 44 #173.
9. A. Avez, *Propriétés ergodiques des endomorphismes dilatants des variétés compactes*, *C. R. Acad. Sci. Paris Sér. A–B* 266 (1968), A610–A612. MR 37 #6944.
10. K. Krzyzewski and W. Szlenk, *On invariant measures for expanding differentiable mappings*, *Studia Math.* 33 (1969), 83–92. MR 39 #7067.

DEPARTMENT OF MATHEMATICS, JAGELLONIAN UNIVERSITY, CRACOW, POLAND

INSTITUTE FOR FLUID DYNAMICS AND APPLIED MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742