Existence of Solutions of Two-Point Boundary Value Problems for Nonlinear Systems

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1. We study two-point boundary-value problems for the nonlinear equation

\[ x'' = f(t, x, x') \] (1.1)

where \( f : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is continuous. Write \( X \) for \([0, 1] \times \mathbb{R}^d \times \mathbb{R}^d\).

By far the large majority of results on the existence of solutions of two-point boundary value problems have been for the one-dimensional case, \( d = 1 \). Papers on nonlinear higher dimensional cases include [5, 6, 8, 10 and 11].

Consider the boundary condition

\[ x(0) = 0, \quad x(1) = 0, \] (1.2)

or, more generally, where \( A_0 \) and \( A_1 \) are \( d \times d \) matrices,

\[ x(0) - A_0 x'(0) = 0, \] (1.3)

\[ x(1) + A_1 x'(1) = 0. \] (1.4)

Let \( \langle \cdot, \cdot \rangle \) denote the inner product and \( |x| \) will denote the Euclidean norm. For a matrix \( A \) we will say \( A \) is positive definite and write \( A > 0 \) if \( x \cdot (A x) > 0 \) for all \( x (x \neq 0) \) in \( \mathbb{R}^d \). We say that \( A \geq 0 \) if either \( A > 0 \) or \( A \) is identically 0.

We assume in our theorems either that \( A_0, A_1 \geq 0 \) or that \( A_0, A_1 > 0 \).

Our two-point boundary conditions are analogs of the boundary conditions studied in dimension \( d = 1 \) by Keller [9] and Bebernes and Gaines [2], (and the other conditions of our Theorem 3 generalize their conditions). On the other hand, problems (1.1), (1.3), and (1.4) include (by letting \( A_0 = A_1 = 0 \)) the classical two-point boundary value problems (1.1) (1.2)

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which were studied in $d$-dimensions by Scorza–Dragoni [10, 11] and recently by Hartman [6] and Hermes [8].

We investigate the condition on $f$ that there does exist a $K > 0$ and $\sigma > 0$ such that one of the following holds

\[
|y|^2 + x \cdot f(t, x, y) \geq -K(1 + |x| + |x \cdot y|) + \sigma |y|,
\]
for $(t, x, y) \in X$, (1.5)

\[
|y|^2 + x \cdot f(t, x, y) \geq -K(1 + |x| + |x \cdot y|) + \sigma |f(t, x, y)|,
\]
for $(t, x, y) \in X$, (1.6)

either of which implies

\[
|y|^2 + x \cdot f(t, x, y) \geq -K(1 + |x| + |x \cdot y|),
\]
for $(t, x, y) \in X$. (1.7)

Conditions of this type are motivated by examining the "amplitude" $u(t) = \frac{1}{2} |x(t)|^2$ of a solution $x(t)$ of (1.1). The second derivative $u''(t)$ satisfies

\[
u''(t) = |x'(t)|^2 + x(t) \cdot x''(t) = |x'(t)|^2 + x(t) \cdot f(t, x(t), x'(t)),
\]
which is the left hand side of the inequalities (1.5)–(1.7).

Our method then is to use inequalities on the scalar $u(t)$ rather than on $x(t)$. Notice that (1.7) is a condition involving $u$, $u'$, and $u''$, and is essentially

\[
u'' \geq -K(1 + |u|^{1/2} + |u'|).
\]
(1.7a)

The additional terms in (1.5) and (1.6) enable us to then obtain bounds on $|x'|$ after we have bounded $u$ and $|u'|$. The inequality (1.7a) might be generalized or put in more abstract form, but our objective is only to present methods for proving existence of a solution using a particular inequality of substantial generality. After showing $u$ and $|u'|$ are bounded, the methods take on a flavor of the differential geometry of curves, particularly in Lemma 3.

We will prove the following theorems.

**Theorem 1.** Assume $f : X \to \mathbb{R}^d$ is continuous. Assume the following conditions are satisfied.

For all $\lambda \in (0, 1]$ and for each solution $x_0(t)$ of

\[
x'' = \lambda f(t, x, x')
\]
(1.1a)
either $x_0(t)$ is defined for all $t \in [0, 1]$ or $|x_0(t)|$ is not bounded.

For some $K > 0$, (1.7) is satisfied.

\[
A_0 > 0, A_1 > 0, \text{and either } A_0 > 0 \text{ or } A_1 > 0.
\]
(1.10)

Then there is a solution of (1.1) which satisfies (1.3) and (1.4).
SOLUTIONS OF TWO-POINT BOUNDARY VALUE PROBLEMS

This theorem is relatively simple but the reader is not told how to verify whether (1.9) is satisfied. The next two theorems fill this gap. We do not know whether Theorem 1 is true in the case of $A_0 = A_1 = 0$.

**Theorem 2.** Let $f: \mathbb{R}^d \to \mathbb{R}^d$ be continuous. Assume that for some $\alpha > 0$ and $K > 0$, $f$ satisfies (1.6). Assume $A_0 \geq 0$ and $A_1 \geq 0$. Then there is a solution of (1.1) which satisfies (1.3) and (1.4).

The above theorem was proved by Hartman [6, Theorem 21] in the case $A_0 = A_1 = 0$ with a more restrictive form of (1.6)

$$|y|^2 + x \cdot f(t, x, y) \geq -K + \sigma |f(t, x, y)|$$

for all $(t, x, y) \in \mathbb{R}^d$.

He assumed in addition that there is a function $\phi: [0, \infty) \to (0, \infty)$ such that

1. $\int_0^\infty (s/\phi(s))ds = \infty$,
2. $|f(t, x, y)| \leq \phi(|y|)$.

We will say that $\phi$ is a Nagumo function for $f$ if (1.12) and (1.13) are satisfied. The simplest function satisfying (1.12) is the function $\phi(s) = k(1 + s^2)$ for some $k > 0$. See the results by Bernstein [1, pp. 311].

Our proof is based on different geometric ideas from those of Hartman. We are therefore able to omit the "Nagumo" condition in the case where we assume (1.6). When we replace (1.6) by (1.5), we must assume that a Nagumo function for (1.1) exists.

**Theorem 3.** Let $f: \mathbb{R}^d \to \mathbb{R}^d$ be continuous. Assume that for some $\alpha > 0$ and $K \geq 0$, $f$ satisfies (1.5). Assume in addition that there exists a Nagumo function for $f$. Then there is a solution of (1.1) which satisfies (1.3) and (1.4).

**Remark 1.** Suppose we had started with more general (nonhomogeneous) boundary conditions

$$y(0) - A_0 y'(0) = r_0, \quad (1.3)'$$
$$y(1) + A_1 y'(0) = r_1, \quad (1.4)'$$

for given vectors $r_0, r_1$ in $\mathbb{R}^d$ and the equation

$$y'' = g(t, y, y'), \quad (1.1)'$$

where $A_0 \geq 0$ and $A_1 \geq 0$. We could then have reduced this problem to the homogeneous problem (1.3) (1.4) (1.1) by a linear substitution $x = y + v_1 + v_2 t$, where $v_1$ and $v_2$ are to be chosen (in $\mathbb{R}^d$).
Remark 2. Instead of \( \sigma \) being just a constant, our proof shows \( \sigma \) in (1.5) and (1.6) can be any positive continuous function \( \sigma(u, u') \) but \( \sigma \) cannot depend directly on \( |x'| \). That is, \( \sigma \) can depend on \( |x|^2 \) and on \( |x \cdot y| \) but not on \( |y| \). Similarly, it is sufficient to assume that for each compact \( x \) set there is a function \( \phi \) which satisfies (1.12) and (1.13). We could also have made conditions (1.3) and (1.4) more general so as to have completely generalized the one-dimensional conditions of Keller and Bebernes–Gaines, [9, 21]; however, we felt that attempts to fully generalize their conditions would have led to too many cases, and we hope our conditions are a fair compromise. Compare these boundary conditions with those in [13].

2. The proofs of our results use the following lemma which is easily derived from the Leray–Schauder theory of topological degree. The reader might find [12] interesting. Let \( \lambda \) be in \([0, 1]\) and let \( S(\lambda) \) be the set of \( C^2 \) functions \( x : [0, 1] \rightarrow \mathbb{R}^d \) which satisfy

\[
x'' = \lambda f(t, x, x')
\]

and satisfy (1.3) and (1.4).

**Lemma 1.** Assume \( A_0 \geq 0 \) and \( A_1 \geq 0 \). Suppose there is a constant \( B > 0 \) such that if \( \lambda \) is in \([0, 1]\) and \( x(\cdot) \) is in \( S(\lambda) \), then

\[
| x(t) | \leq B \quad \text{and} \quad | x'(t) | \leq B \quad \text{for all } t \text{ in } [0, 1].
\]

Then \( S(1) \) is not empty.

Observe that there is a unique solution (namely, \( x \equiv 0 \)) of the homogeneous equation

\[
x'' = 0
\]

which satisfies (1.3) and (1.4) when \( A_0, A_1 \geq 0 \). In fact, the general solution of (2.3) is \( x(t) = v_1 + v_2 t \) and (as in Remark 1) \( v_1 \) and \( v_2 \) are uniquely determined as linear functions of \( r_1 \) and \( r_2 \) so we must have \( v_1 = 0 = v_2 \). Therefore (as in [3]) there exists a Green's function (matrix) \( \Gamma(t, s) \) such that Eq. (2.1) with conditions (1.3) and (1.4) is equivalent to

\[
x(t) = \lambda \int_0^1 \Gamma(t, s) f(s, x(s), x'(s)) \, ds.
\]

The map

\[
x(t) \rightarrow \int_0^1 \Gamma(t, s) f(s, x(s), x'(s)) \, ds
\]
is completely continuous in the space $C_{[0,1]}^1$ of continuously differentiable functions on $[0, 1]$ with the supremum norm. The proof follows immediately by applying the Leray–Schauder alternative. See Granas [4].

Our second lemma is needed to show that $|x(t)|$ and the radial component of $x'$ (or rather $u$ and $|u'|$) are bounded. The bound $R_0$ described below is independent of the particular solution $u$ and depends only on $\alpha$ and $K$.

**Lemma 2.** Let $K$ and $\alpha$ be nonnegative constants. Let $u(t)$ be a nonnegative function satisfying the inequalities

\begin{align*}
  u'(0) &\geq 0, \quad u'(1) \leq 0, \\
  u(0) &\leq \alpha u'(0), \\
  u''(t) &\geq -K[1 + (2u(t))^{1/\alpha} + |u'(t)|].
\end{align*}

Then there exists a constant $B_0$ (depending on $K$ and $\alpha$ only) such that

\begin{align*}
  u(t) &\leq B_0, \\
  |u'(t)| &\leq B_0 \quad \text{for all } t \in [0, 1].
\end{align*}

In the proof we show that

\begin{align*}
  |u(t)| &\leq 3\beta^{-1}e^{3(K+1)}, \\
  |u'(t)| &\leq 4\beta^{-1}e^{4(K+1)} \quad \text{for } 0 \leq t \leq 1
\end{align*}

where $\beta = \min\{1, \alpha\}$. Hence we may let

\begin{equation}
  B_0 = 4\beta^{-1}e^{4(K+1)}.
\end{equation}

**Proof.** We shall use in the proof the following inequality

\begin{equation}
  \int_0^8 \frac{dz}{a + kx + x^2} > \int_0^8 \frac{dz}{a + x(k + 1)} > 1 \quad \text{when } a \leq \beta ke^{-k-1}, \beta \leq 1
\end{equation}

($a, \beta, k$ nonnegative) which we leave to the reader. Now let us assume that the maximum of $u(t)$ occurs at a point $t_1$. From the conditions (2.4) it follows immediately that $u'(t_1) = 0$. When $t_1 = 0$ we have $\max u(t) = u(t_1) \leq \alpha u'(t_1) = 0$ and consequently $u \equiv 0$. Therefore we assume that $t_1 > 0$. Define

\begin{equation}
  L = \frac{3}{2\beta} e^{3(K+1)}.
\end{equation}

There are two cases under consideration either $u(t_1) < 2L$ or $u(t_1) > 2L$. Assume the first one. We have $(2u)^{1/2} \leq (4L)^{1/2} \leq 2L$ and

\begin{equation}
  u'' \geq -K(1 + 2L + |u'|).
\end{equation}
Therefore the function \( w = -u' \) satisfies the first-order differential inequality
\[
|w'| \leq +K(1 + 2L) + K|w|
\]

Because \( w(0) \leq 0 \) [from (2.4)], we can apply the standard technique of first-order differential inequalities to obtain \( w(t) \leq (1 + 2L)(e^K - 1) \) for all \( t \in [0, 1] \). And since \( w(1) \geq 0 \) [from (2.4)] we obtain
\[
w(t) \geq -(1 + 2L)(e^K - 1);
\]
that is,
\[
|u'| = |w| \leq (1 + 2L)(e^K - 1).
\]

From this and the definition of \( L \), the second of the inequalities (2.8) follows easily. The first follows from the assumption that \( \max u(t) = u(t_1) \leq 2L \).

Thus in order to finish the proof, it is enough to show that the second case \( u(t_1) > 2L \) is impossible. Suppose \( u(t_1) > 2L \). Define \( t_0 = 0 \) if \( u(t) > L \) for \( t \in [0, t_1] \). If not define \( t_0 = \sup \{ t \in [0, t_1] : u'(t) \geq \frac{1}{2}u(t) \} \). Since \( u(t_1) > 2L \), by the mean value theorem \( t_0 \) is well defined. It is easy to observe that
\[
(2.11)
\]

For any positive \( h \) we have \( (2u)^{1/2} \leq hu + \frac{1}{2}h^{-1} \). Thus the inequality (2.6) implies
\[
u'' \geq -K(1 + \frac{1}{2}h^{-1} + hu + |u'|).
\]
Setting \( z = u'u^{-1} \) and \( a = KL^{-1}(1 + \frac{1}{2}h^{-1}) + hK \), we have in the interval \([t_0, t_1]\)
\[
z' \geq -a - K|z| - z^2.
\]

Thus
\[
-\int_{z(t_0)}^{z(t_1)} \frac{dz}{a + K|z| + z^2} \leq t_1 - t_0 \leq 1.
\]

Since \( z(t_1) = 0 \) and from (2.11) \( z(t_0) \geq \beta \), we have
\[
\int_{0}^{\beta} \frac{dz}{a + K|z| + z^2} \leq 1.
\]

Setting \( h = \frac{1}{2}Ke^{-K-1} \) it is easy to verify that \( a < \beta Ke^{-K-1} \) which contradicts (2.10).
Remark 3. We now show that the hypotheses of Theorems 1–3 imply that the hypotheses of Lemma 2 (and hence its conclusions) are true. Define $S$ to be $\{(\lambda, x'(\cdot)) : \lambda \in [0, 1] \text{ and } x \in S(\lambda)\}$. We know $S$ is not empty since if $x \equiv 0$, then $(0, x) \in S$. Choose $(\lambda, x) \in S$ and define $u(t) = \frac{1}{2} | x(t) |^2$. Then from the fact that $A_0 > 0$ in (1.3) and $A_1 > 0$ in (1.4) we have

$$
\begin{align*}
    u'(0) &= x'(0) \cdot x(0) = x'(0) \cdot A_0 x'(0) \geq 0, \\
    u'(1) &= x'(1) \cdot x(1) = x'(1) \cdot [-A_1 x'(1)] \leq 0;
\end{align*}
$$

so (2.4) is satisfied. Since $A_0 > 0$, either $A_0 = 0$ or $A_0$ is positive definite. If $A_0$ is 0, then $u(0)$ must be 0 and (2.5) is satisfied with $\alpha = 0$. If $A_0$ is positive definite, we can define $\alpha = \frac{1}{2} \sup \{ |A_0 y|^2 / (y \cdot A_0 y) : |y| = 1 \}$. Let $x_0 = x(0)$ and $y_0 = x'(0)$ so that $x_0 = A_0 y_0$. Then $u(0) = \frac{1}{2} | x_0 |^2 = \frac{1}{2} | A_0 y_0 |^2 \leq \alpha y_0 \cdot A_0 y_0 = \alpha u(0)$, verifying (2.5). Furthermore, since $(\lambda, x) \in S$, $u$ satisfies (2.6), which is implied by (1.5), by (1.6), and by (1.7).

3. We now prove Theorems 1, 2 and 3.

Proof of Theorem 1. From Remark 3, Lemma 2 implies there is a positive $B_0 (= B_0(\alpha, K))$ such that for each $(\lambda, x) \in S$ (letting $u = \frac{1}{2} | x |^2$)

$$
u(t) \leq B_0, \quad |u'(t)| \leq B_0 \quad \text{for all } t \in [0, 1]. \tag{3.1}$$

Let $C^1$ be the set of continuously differentiable functions $\phi : [0, 1] \to \mathbb{R}^d$ with $\| \phi \|_1 = \max \{ \| \phi \|, \| \phi' \| \}$ where $\| \phi \| = \sup_{[0,1]} |\phi|$. We claim $S$ is compact subset of $[0, 1] \times C^1$. Suppose there exists a sequence $\{(\lambda_i, x_i(\cdot))\}$ with no limit points in $S$. We may assume $\lambda_i$ and $x_i(0)$ (since $|x_i(0)|^2 \leq 2B_0$) are convergent, so we now assume $\lambda_0$ and $x_0$ are points such that $\lambda_i \to \lambda_0$ and $x_i(0) \to x_0$ as $i \to \infty$. Let $y_i = x_i'(0)$ for all $i$. Suppose $A_0 > 0$. (The case $A_0 = 0$, $A_1 > 0$ is similar and is omitted). Then $y_i = A_0^{-1} x_i(0)$ [from (1.3)], which converges to some $y_0 (= A_0^{-1} x_0)$. From the convergence Theorem, [7, p. 14], there is a solution $x_0(t)$ of (2.1) with $\lambda = \lambda_0$ (defined on an interval $J \subset [0, 1]$ with $0 \in J$) such that (i) $x_0(0) = x_0$ and $x_0'(0) = y_0$ and either $J = [0, 1]$ or $J = [0, T)$ for some $T \in (0, 1]$ (and $x_0(t)$ cannot be defined continuously on any larger interval or even at $T$); (ii) this solution has the property that

$$
x_i(t) \to x_0(t) \quad \text{and} \quad x_i'(t) \to x_0'(t) \quad \text{for all } t \in J. \tag{3.2}
$$

But since $|x_i(t)|^2 \leq 2B_0$ for all $i$ and $t$ we must have $|x_0(t)| \leq (2B_0)^{1/2}$ for all $t \in J$. Since $x_0(t)$ is bounded, (1.9) implies $J = [0, 1]$. Applying (3.2) at $t = 1$, we get (1.4); so $(\lambda_0, x_0(\cdot)) \in S$, contradicting our assumption that $\{(\lambda_i, x_i)\}$ had no limit points in $S$. Therefore $S$ is compact.
Since \( S \) is compact there is some \( B_1 > 0 \) such that
\[
|x'(t)| \leq B_1 \quad \text{for all } t \in [0, 1] \text{ and all } (\lambda, x) \in S.
\]
Let \( B = \max\{(2B_0)^{1/2}, B_1\} \). Then (2.2) is satisfied. Applying Lemma 1, \( S(1) \) is nonempty; that is, there is a solution of (1.1) satisfying (1.3) and (1.4).

**Proof of Theorem 2.** Let \( B_0 \) be given by (2.9). Let \((\lambda, x)\) be in \( S \) and let \( u(t) = \frac{1}{2} |x(t)|^2 \). Let \( \xi \) be \( K(1 + [2B_0]^{1/2} + B_0) \) so that from (1.6) and \( B_0 \geq |u'| \), we have for all \( t \in [0, 1] \),
\[
u''(t) \geq -\xi + \sigma |x''(t)|;
\]

hence for \( 0 \leq \tau \leq t \leq 1 \).
\[
2B_0 \geq u'(t) - u'(\tau) \geq \int_{\tau}^{t} [-\xi + \sigma |x''(s)|] \, ds \quad \text{for } 0 \leq \tau \leq t \leq 1
\]
\[
\geq -\xi + \sigma \int_{\tau}^{t} |x''(s)| \, ds \geq -\xi + \sigma \int_{\tau}^{t} x''(s) \, ds
\]
\[
\geq -\xi + \sigma (|x'(t)| - |x'(\tau)|).
\]
Integrating this gives
\[
(2B_0 + \xi) \sigma^{-1} = \int_{0}^{1} (2B_0 + \xi) \sigma^{-1} \, d\tau \geq \int_{0}^{1} |x'(t) - x'(\tau)| \, d\tau
\]
\[
\geq \int_{0}^{1} |x'(t) - x'(\tau)| \, d\tau \geq |x'(t) - [x(1) - x(0)]|
\]
\[
\geq |x'(t)| - |x(1)| - |x(0)|;
\]

hence
\[
(2B_0 + \xi)\sigma^{-1} + 2(2B_0)^{1/2} \geq |x'(t)| \quad \text{for all } t \in [0, 1]. \tag{3.3}
\]
Letting \( B \) be the left-hand side of (3.3), the hypotheses of Lemma 1 are satisfied, so there is a solution of (1.1) satisfying (1.3) and (1.4).

The existence of a Nagumo function is a weak assumption if \( d > 1 \). We cannot conclude anything about the boundedness of \( |x| \) and \( |x'(t)| \). Instead we are able to make conclusions on the arc length of the curve \( x(t) \), that is \( \int |x'| \).

**Lemma 3.** Let \( f : X \to \mathbb{R}^d \) be continuous. Let \( \phi \) be a Nagumo function for \( f \) (satisfying (1.12) and (1.13)). Let \( \gamma : [0, \infty) \to [0, \infty) \) be the function defined by
\[
\int_{\theta}^{\gamma(t)} s \, ds = \frac{\theta}{\phi(s)} \quad \text{for } \theta \in [0, \infty). \tag{3.4}
\]
Let $x$ be a solution of (1.1) defined on $[0, 1]$ and let $\theta_1$ be the arc length of $x$; i.e.,

$$\theta_1 = \int_0^1 |x'(s)| \, ds.$$ 

Then $|x'(t)| \leqslant \gamma(\theta_1)$ for all $t \in [0, 1]$.

Equation (1.12) guarantees that $\gamma$ is defined for all $\theta \geqslant 0$ since $\int_0^\infty \phi(s)^{-1} \, ds = \infty$ so there is some number $\gamma(\theta)$ for which (3.4) is satisfied.

Proof of Lemma 3. From the inequality (1.13),

$$|x'(s)| \geqslant |x'(\bar{s})| \frac{|x''(\bar{s})|}{\phi(|x'(\bar{s})|)} \geqslant |x'(\bar{s})| \frac{|x''(\bar{s})|}{\phi(|x'(\bar{s})|)}.$$ 

Hence if $0 \leqslant v < 1, 0 \leqslant w < 1,$

$$\theta_1 \geqslant \left| \int_v^w \frac{x' \cdot x''}{\phi(|x'|)} \, ds \right| = \left| \int_{|x'(v)|}^{|x'(w)|} s \, ds \phi(s) \right| (3.5)$$

by the "change of variables" $|x'(s)| \to s$; the two integrals are equal (and 0) for $w = v$, and the derivatives with respect to $w$ exist almost everywhere and are equal. It follows that the integrals are equal since $|x'(w)|$ is absolutely continuous. Choose $v$ and $w$ such that $|x'(v)| = \theta_1$ (which is possible by the mean-value theorem) and $|x'(w)| = \max |x'(s)|$ on $0 \leqslant s \leqslant 1$. The desired result then follows from (3.4) and (3.5).

Proof of Theorem 3. Let $B_0$ and $(\lambda, x)$ and $u$ and $\xi$ be as in the proof of Theorem 2. Let $\theta(\xi) = \int_0^1 |x'(s)| \, ds$. From (1.5) and $B_0 \geqslant |u'|$, we have for all $t \in [0, 1],

$$u''(t) \geqslant -\xi + \sigma |x'(t)|$$

$$2B_0 \geqslant u'(1) - u'(0) \geqslant \int_0^1 [-\xi + \sigma |x'(s)|] \, ds = -\xi + \sigma \theta(1),$$

so the arc length $\theta(1)$ satisfies $\theta(1) \leqslant \xi_1$, where $\xi_1 = (2B_0 + \xi)/\sigma$. From Lemma 3,

$$|x'(t)| \leqslant \gamma(\xi_1) \quad \text{for all } t \in [0, 1],$$

where $\gamma$ is given by (3.4). Letting $B = \max\{(2B_0)^{1/\alpha}, \gamma(\xi_1)\}$, the hypotheses of Lemma 1 are satisfied, so there is a solution of (1.1) satisfying (1.3) and (1.4).
REFERENCES