Asymptotic Stability for one Dimensional Differential-Delay Equations*

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Revised December 17, 1968

1. INTRODUCTION

Let $C^q$ be the set of continuous functions $\phi : [-q, 0] \rightarrow \mathbb{R}$ where $q \geq 0$, and let $\|\phi\| = \sup_{s \in [-q, 0]} |\phi(s)|$ for $\phi \in C^q$. The stability definitions are given in the next section. Let

$$C^q_\beta = \{\phi \in C^q : \|\phi\| < \beta\}.$$ 

If $x(t)$ is defined and continuous on $[t - q, t]$, we will write $x_t$ for the function for which $x_t(s) = x(t + s)$ for $s \in [-q, 0]$. Hence $x_t \in C^q$.

This paper shows that for a nonlinear one-dimensional differential delay equation

$$x'(t) = F(t, x_t(\cdot)) \quad \text{(DDE)}$$

one can frequently determine (almost by inspection) if the 0 solution is asymptotically stable and give a region of attraction. Theorem 1.1 gives a simple, readily applicable criterion for asymptotic stability. No use is made of complicated criteria such as the existence of a Liapunov function. For $\phi \in C^q$, define

$$M(\phi) = \sup_{t \in [0, \infty)} \sup_{s \in [-q, 0]} \phi(s).$$

1.1. THEOREM. Let $\beta > 0$ and $q > 0$. Let $F : [0, \infty) \times C^q_\beta \rightarrow \mathbb{R}$ be continuous. Assume for some $\alpha > 0$

$$\alpha M(\phi) \geq -F(t, \phi) \geq -\alpha M(-\phi) \quad \text{for all } \phi \in C^q_\beta. \quad (1.1)$$

(i) Assume $\alpha q \leq \frac{\beta}{2}$. Then $x(t) \equiv 0$ is a solution and is uniformly stable.

(ii) Assume $0 < \alpha q < \frac{\beta}{2}$ and

for all sequences $t_n \rightarrow \infty$ and $\phi_n \in C^q_\beta$ converging to a constant nonzero function in $C^q_\beta$, $F(t_n, \phi_n)$ does not converge to 0. \quad (1.2)
Then 0 is uniformly-asymptotically stable, and if
\[ \| x_{t_0} \| \leq 2\beta/5 \]

for any \( t_0 \geq 0 \), then \( x(t) \to 0 \) as \( t \to \infty \).

Remark. The uniform stability and uniform asymptotic stability of the zero solution can be made more specific. The proof shows that if \( \alpha q \leq \frac{3}{2} \):

If \( \| x_{t_0} \| < 2\beta/5 \) for any \( t_0 \geq 0 \), then the solution \( x(t) \) is defined and satisfies \( |x(t)| < 5\| x_{t_0} \|/2 \) for all \( t \geq t_0 \); \hspace{1cm} (1.4)

\[ V(t) \overset{\text{def}}{=} \sup_{t \leq s \leq t + 3\alpha} |x(s)| \]

is a monotonic non-increasing function for \( t \geq t_0 \);

If \( 0 < \alpha q < \frac{3}{2} \), then \( V(t) \to 0 \). \hspace{1cm} (1.5)

A special case of (1.1) is the more intuitive but much more restrictive condition \( \alpha \sup \phi \geq -F(t, \phi) \geq \alpha \inf \phi \), which is satisfied by the right-hand sides of the equation
\[ \dot{x}(t) = -\alpha x(t - r(t)), \]

where \( \alpha \geq 0 \) is constant and \( r \) is continuous and nonnegative in (L), that is
\[ F(t, x_t) = -\alpha x(t - r(t)), \]
i.e., \( F(t, \phi) = -\alpha \phi(-r(t)) \).

If we want to force \( x(t) \) to 0 as \( t \to \infty \), a natural condition to require of \( F \) is that for each \( t \) and \( \phi \), \( F(t, \phi) \) has opposite sign from \( \phi(s) \) for some \( s < 0 \), (i.e. \( F(t, \phi) \phi(s) \leq 0 \)). Condition (1.1) includes this criterion but it also requires that the control response \( F(t, \phi) \) cannot be too strong. It is common experience that too strong a control is unstable. Here our criterion of strength depends on the size of the lag and the size of \( |F| \).

The requirement "\( \alpha q \leq \frac{3}{2} \)" prevents a solution from oscillating with oscillations growing larger and larger. Condition (1.2) on the other hand is designed only to insure that a solution which monotonically approaches a constant in fact goes to 0 as \( t \to \infty \). If \( F \) is autonomous (i.e., \( F(t, \phi) = F(\phi) \) so \( F \) is independent of \( t \), from continuity of \( F \) it is readily seen that (1.2) is equivalent to:

if \( x(t) \equiv \gamma \) is a solution of (DDE) and \( |\gamma| < \beta \), then \( \gamma = 0 \). \hspace{1cm} (1.8)

Theorem 1.1 implies the asymptotic stability of (DDE) for many standard equations. Consider the following observations. The linear equation with constant coefficients
\[ \dot{x} = -\alpha x(t - q) \]
with \( \alpha \geq 0 \) and \( q \geq 0 \) constant satisfies (1.1). It is well known that 0 is stable if and only if \( 0 \leq \alpha q \leq \pi/2 \approx 1.57 \) and is asymptotically stable for
0 < \alpha q < \pi/2. Theorem 1.1 requires \alpha q \leq 1.5 for stability. Therefore even when (1.7) reduces to the constant coefficients case (1.9) the constant \beta cannot be substantially improved. For nonconstant coefficients the constant \beta cannot be improved (i.e. increased) at all because examples in [2], [3], and [4] show for \alpha q = \frac{\pi}{2}, (1.7) can have a nontrivial periodic solution.

If \alpha(t) > \alpha_0 > 0 and is bounded, if \gamma > 0 is arbitrary and if \gamma > 1 is an odd integer, then Theorem 1.1 says that 0 is asymptotically stable for

\[ \dot{x} = -\alpha(t) x(t - r(t)) \]  

where \( r(t) \in [-q, 0] \). If on the other hand \gamma is an even integer, 0 is not asymptotically stable even for the ordinary differential equation case with \( r(t) \equiv 0 \). Also, \( x(t) \equiv 0 \) can be shown to be unstable if \( \gamma = \frac{\pi}{2} \) (and in fact for the case \( \alpha(t) = \text{constant}, r(t) = q \), it will be proved in a future paper by the author that there exists a nontrivial orbitally asymptotically stable solution of period \( 4\gamma \), and hence this case is not covered by the theorem.

The research in this paper was stimulated by a "Research Problem" of Bellman [1] concerning (1.7). When the proof (given in Section 4) was first found for (1.7), it was then obvious that the proof extended to a much larger class of equations, but it has been much more difficult to find a clear and general statement of the theorem (now embodied in conditions (1.1) and (1.2)) than to find the proof. Some additional results and a primitive version of Theorem 1.1 are stated without proof in [2]. (See in particular Theorem 2 in [2].)

The case in which \( F(t, \phi) \) is linear in \( \phi \) was studied by Myskis [3] and Lillo [4]. Myskis showed that if \( \alpha q \leq \frac{\pi}{2} \), each solution is bounded. (J. Kato informed me that the boundedness of each solution of a linear equation implies 0 is stable; the proof is easy using the Banach Steinhauss theorem.) Myskis showed that if \( 0 < \alpha q < \frac{\pi}{2} \), each solution tends to 0 as \( t \to \infty \). An example of Myskis shows that if \( \alpha q > \frac{\pi}{2} \), a continuous linear \( F \) can be constructed in which some solutions are unbounded. It is well known that if \( F(t, \phi) \) is continuous and is linear, then \( F(t, \phi) = \int_{-q}^{0} \phi(s) \, d\eta(t, s) \), a Lebesgue-Stieltjes integral for each \( t \). Then (1.1) is satisfied if and only if for each \( t, \eta(t, \cdot) \) can be chosen non-increasing and \( -\int_{-q}^{0} \, d\eta(t, s) \leq \alpha \).

Extensions and examples of Theorem 1.1 are given in Section 3. There we consider multi-dimension systems and equations for which 0 is not a solution but each solution is stable. The proof of Theorem 1.1 is given in Section 4.

2. Notation

We say \( x \) is a solution on \([t_0, t_1]\) of the differential-delay equation (DDE) if \( \psi \) is a continuous real-valued function, defined on an interval \([t_0 - q, t_1]\), where \( \infty \geq t_1 > t_0 \geq t_0 - q \), and satisfies (DDE) on \((t_0, t_1)\), and we write
\[ x(t) = x(t; t_0, \phi) \text{ where } \phi = x(t_0). \] We will also say such a solution is a solution at \( t_0 \) if \( x \) is a solution on \([t_0, t]\) for some \( t \geq t_0 \).

2.1. Definition. We say \( 0 \) is uniformly stable for (DDE) if for any \( \eta > 0 \) there exists a \( \delta = \delta(\eta) \) in \((0, \eta]\) such that for any \( t_0 > 0 \) and \( \phi \in C^0 \) and any solution \( x = x(t; t_0, \phi) \) we have for all \( t > t_0 \) in the domain of \( x \)

\[ 0 < \|\phi\| < \delta \quad \text{implies} \quad \|x(t; t_0, \phi)\| < \eta. \]

It follows that \( x(t) \equiv 0 \) is a solution. Note theorem 2.3 iv.

2.2. Definition. Let \( x \) be a solution on \([t_0, T)\), where \( 0 \leq t_0 < T \leq \infty \)

We say \( x \) is noncontinuable either if \( T = \infty \), or if \( T < \infty \) and for every \( \varepsilon > 0 \), \( x \) cannot be extended to \([t_0, T + \varepsilon)\) in such a way that \( x \) is a solution on \([t_0, T + \varepsilon)\).

Actually if \( x \) is a noncontinuable solution with domain \([t_0, T)\) for \( T < \infty \) and \( F \) is continuous, \( F : [0, \infty) \times C^0_{\beta_0} \to \mathbb{R} \), then either \( \lim_{t \to T} x(t) \) does not exist or \( \lim_{t \to T} x(t) = \beta \). Hence \( x(\cdot) \) cannot be defined at \( T \) so that \( x(T) \in (-\beta, \beta) \) so that \( x \) is continuous at \( T \).

We shall need the following well known results. The proofs are quite similar to the proofs for ordinary differential equations. Note that (iv) follows from (ii) and (iii).

2.3. Theorem. Let \( F : [0, \infty) \times C^0_{\beta_0} \to \mathbb{R} \) be continuous.

(i) For each \( t_0 \geq 0 \) and \( \phi \in C^0_{\beta_0} \), there exists an \( \varepsilon > 0 \) and a solution \( x(\cdot) \) of (DDE) on \([t_0, t_0 + \varepsilon)\) such that \( x(t_0) = \phi \).

(ii) Let \( 0 \leq t_0 < T < \infty \) and let \( x(\cdot) \) be a non-continuable solution on \([t_0, T)\). Then \( x(\cdot) \) has no limit points in \( C^0_{\beta_0} \) as \( t \to T \). If \( \beta_0 < \beta \) and \( \|F\| \) is bounded on \([0, T] \times C^0_{\beta_0} \), then \( \{t \in [t_0, T) : x(t) \in C^0_{\beta_0} \} \) is compact; that is, \( x \) leaves \( C^0_{\beta_0} \) as \( t \) approaches \( T \).

(iii) Any solution on \([t_0, T)\) can be extended to an interval on which it is noncontinuable.

(iv) For some \( \beta > 0 \) let \( \|F\| \) be bounded on \([0, \infty) \times C^0_{\beta_0} \) and let \( 0 \) be uniformly stable. Then there exists \( \delta > 0 \) such that each noncontinuable solution \( x \) satisfying \( \|x(t_0)\| < \delta \) for any \( t_0 \geq 0 \) is defined on \([t_0, \infty)\).

(v) If \( F \) is defined on \([0, \infty) \times C^0 \) and for some \( \alpha \)

\[ |F(t, \phi)| \leq \alpha \|\phi\| \quad \text{for all} \quad t \geq 0, \quad \phi \in C^0, \]

then for \( t_0 \geq 0 \) each noncontinuable solution at \( t_0 \) is defined on \([t_0, \infty)\).
2.4. Definition. Let \( \gamma > 0 \). We say 0 is uniform-asymptotically stable with attraction radius \( \gamma \) (for (DDE)) if

(i) 0 is uniformly stable,

(ii) for each \( t_0 \geq 0 \), each noncontinuable solution \( x \) at \( t_0 \) with \( \| x(t_0) \| \leq \gamma \) has domain at least \([t_0, \infty)\),

(iii) there exists \( T = T(\gamma_1) \) for each \( \gamma_1 \in (0, \gamma) \) such that for each \( t_0 \geq 0 \) and each solution \( x \) of (DDE) with \( \| x(t_0) \| \leq \gamma_1 \)

\[
| x(t_0 + s) | \leq \gamma_1/2 \quad \text{for all} \quad s \geq T(\gamma_1). \tag{2.1}
\]

It follows that if 0 is UAS, then \( \| x(t_0) \| < \gamma \) implies \( x(t) \to 0 \) as \( t \to \infty \).

From now on solution will be assumed to mean noncontinuable solution. Although Theorem 2.3 is stated for one-dimensional equations, the results also are true (using obvious generalizations) for higher dimensional equations (or systems).

3. Corollaries of the Main Theorem

This section presents two applications (Corollaries 3.1 and 3.2) to complicated equations and an extension for higher dimension (Corollary 3.3) and a result for the case in which 0 is not a solution (Corollary 3.4). We say 0 is globally uniform-asymptotically stable (GUAS) for

\[
\dot{x}(t) = -g(x(t - r[t, x(t)])) \tag{3.1}
\]

if 0 is UAS with attraction radius \( \gamma \) for all \( \gamma > 0 \). Hence if we write \( F(t, \phi) = -g(\phi(-r[t, \phi(0)])) \), if \( g \) and \( r \) are continuous, then \( F \) is continuous on \([0, \infty) \times C^q \) and (3.1) becomes (DDE). Since \( F \) is continuous, solutions exist. (3.1) is a non-autonomous equation since \( r \) depends on \( t \).

3.1. Corollary. Let \( g : \mathbb{R} \to \mathbb{R} \) be continuous, with \( ax^2 > xg(x) > 0 \) for some \( a \) and all \( x \neq 0 \). Let \( r : [0, \infty) \times \mathbb{R} \to [0, q] \) be continuous. If \( axq < \frac{q}{2} \), then 0 is GUAS for (3.1).

The equation \( \dot{x} = -ax(t - |x(t)|) \) is not an example of (3.1) since \( r(t, \phi(0)) = |\phi(0)| \) is not bounded. Also 0 is not globally asymptotically stable for \( \alpha \neq 0 \) since if \( t_0 = 0 \) and \( x : \mathbb{R} \to \mathbb{R} \) satisfies \( x(-1) = -\alpha^{-1} \) and \( x(t) - t + 1 \) for \( t \geq 0 \), then \( x \) is an unbounded solution on \([0, \infty)\); however, 0 is (locally) UAS for \( \alpha > 0 \) since if \( F(\phi) = -a\phi(-|\phi(0)|) \), then \( F : C_0^q \to \mathbb{R} \) for \( q \geq \beta > 0 \) and the conditions of Theorem 1.1 are satisfied if \( 0 < \alpha\beta < \frac{q}{2} \) (letting \( \beta = q \)).
Proof of Corollary 3.1. For every \( \beta > 0 \), \( \rho g(\rho) < \alpha \rho^2 \) for \( 0 < |\rho| < \beta \).

Fix \( \phi = x_t \). Let \( \rho = x(t - r[t, x(t)]) = x_t(-r[t, x_t(0)]) = \phi(-r[t, \phi(0)]) \).

Then if \( \rho > 0 \), \( g(\rho) \leq \alpha \rho \) and

\[
\alpha M(\phi) = \alpha \sup \phi \geq \alpha \rho \geq g(\rho) = F(t, \phi) \geq 0 \geq \alpha M(\phi),
\]

so (1.1) is satisfied. Similarly if \( \rho \leq 0 \), (1.1) is satisfied. If \( t_n \to \infty \) and \( \phi_n \to \) constant \( c \neq 0 \), then for \( n \) sufficiently large

\[ g(\phi_n(-r[t_n, \phi_n(0)])) \to g(c) \neq 0 \quad \text{as} \quad n \to \infty \]

so (1.2) is satisfied and 0 is GUAS.

Corollary 3.2 shows how Theorem 1.1 can be applied to another kind of equation. Here \( F \) is given in terms of a Stieltjes integral. Let

\[ F(t, \phi) = F(\phi) = -\int_0^q g(s, \phi(-s)) \, d\eta(s). \quad (3.2) \]

Hence by choosing \( \eta \) piece-wise constant with discontinuities at \( r_i \), of size \( c_i \), an example of \( F(t, x_t) \) in (3.2) is the multiple lag case

\[ F(t, x_t) = \sum_{i=1}^n c_i g_i(x(t - t_i)) \]

where \( 0 \leq r_1 < \cdots < r_n = q \), \( c_i > 0 \), and \( g_i(x) = g(r_i \cdot x) \) for \( i = 1, \ldots, n \), and \( \sum_{i=1}^n c_i = 1 \). In (3.2) \( F \) is autonomous since \( g \) does not depend on \( t \in \mathbb{R} \), even though \( g \) does depend on \( s \in [-q, 0] \). When \( g \) is continuous, \( F \) is continuous on \( C_\alpha \) since if \( \phi_n \to \phi \in C_\alpha \), \( F(\phi_n) \to F(\phi) \).

3.2. Corollary. Let \( F : \mathbb{R} \times C_\alpha \to \mathbb{R} \) be given by (3.2) where \( q > 0 \) and \( \eta \) is monotonically increasing on \([0, q]\) and \( \eta(q) - \eta(0) = 1 \). Let \( g : [0, q] \times \mathbb{R} \to \mathbb{R} \) be continuous. Assume for some \( \alpha \), \( 0 < \alpha q < \frac{q}{2} \), and for some \( \beta > 0 \)

\[ 0 < xg(s, x) \leq \alpha x^2 \quad \text{for} \quad s \in [0, q] \quad \text{and} \quad 0 < |x| < \beta. \quad (3.3) \]

Then 0 is UAS for,

\[ \dot{x} = F(x_t) = -\int_0^q g(s, x(t - s)) \, d\eta(s). \quad (3.4) \]

Note that if (3.4) is linear, then \( g(s, x) = a(s) \cdot x \) and (3.3) can be changed to

\[ 0 \leq a(s) \quad \text{and} \quad 0 \leq \int_0^q a(s) \, d\eta(s) \leq \alpha. \]
Proof. Choose \( \phi \in C^q_\alpha \). We may assume \( F(\phi) \leq 0 \) since the proof for \( F(\phi) > 0 \) is similar. Let \( E = \{ s \in [0, q] : \phi(-s) \geq 0 \} \).

\[
-F(\phi) \leq \int_E g(s, \phi(-s)) \, d\eta(s) \leq \int_E \alpha \phi(s) \, d\eta(s) \leq \alpha M(\phi).
\]

Since \( -F(\phi) \geq 0 > -M(-\phi) \), \( \alpha M(\phi) \geq -F(\phi) \geq -\alpha M(-\phi) \) and (1.1) is satisfied. Since \( F \) is independent of \( t \), to show (1.2) it suffices to show (1.8). If \( \phi(s) \equiv c \neq 0 \), (3.3) implies

\[
F(\phi) = -\int_0^a g(s, c) \, d\eta(s) \neq 0
\]

so (1.2) holds. Hence by Theorem 1.1, \( 0 \) is UAS.

The main theorem can be immediately generalized to higher dimensions, though the result is not very general. The question of a more general approach for higher dimensions remains open. Consider

\[
\dot{x}(t) = F_i(t, x_1, ..., x_d), \quad i = 1, ..., d \tag{3.6}
\]

for any integer \( d \geq 1 \). Write \( C^q_\beta \times \cdots \times C^q_\beta \) for the product \( C^q_\beta \times \cdots \times C^q_\beta \) (\( d \) times).

3.3. Corollary. Let \( \beta, q > 0 \) and let \( F_i : [0, \infty) \times C^q_\beta \to \mathbb{R} \) be continuous for \( i = 1, ..., d \). Assume there exist \( \alpha > 0 \) such that for each \( (\phi_1, ..., \phi_d) \in C^q_\beta \)

\[
\alpha M(\phi_i) \geq -F_i(t, \phi_1, ..., \phi_d) \geq -\alpha M(-\phi_i), \quad i = 1, ..., d. \tag{3.7}
\]

Assume \( \alpha q < \frac{3}{2} \). Assume that (1.2) is satisfied where \( \phi_n \) are in \( C^q_\beta \) instead of \( C^q_\alpha \) and \( F \) is the vector \( (F_1, ..., F_d) \). Then \( 0 \) is UAS for (3.6).

Proof. Let \( F = (F_1, ..., F_d) \). Then one can show

\[
|F(t, x_1, ..., x_d)| = \sup\{|F_1|, ..., |F_d|\} \leq \alpha \sup\{|\phi_1|, ..., |\phi_d|\}
\]

for all \( (\phi_1, ..., \phi_d) \in C^q_\beta \). Since \( F \) is continuous, as in Theorem 2.3 (v), it follows that each solution of (3.3) defined at any \( t_0 \) will be defined for all \( t \geq t_0 \). Let \( x = (x_1, ..., x_d) \) be a noncontinuable solution of (3.6) on \([t_1, T]\). Fix \( i \). Let

\[
F(t, x) = F_i(t, x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_d),
\]

absorbing the given \( d - 1 \) functions into the \( t \) variable of \( F \). Then \( x \) satisfies (DDE). Furthermore, (3.7) implies that (1.1) holds (by letting \( \phi_j = x_j, j \neq i \), and \( x_i = \phi_i \) in (1.4)). From (1.4) \( |x'(t)| \leq 5(2)^{-1} |x_{i_0}'| \) (for each \( i \)). It is now clear \( 0 \) is uniformly stable. As in Theorem 2.3 v if \( |x'(t)| \leq \beta_0 < \beta \) for all
$t \in [t_0, T]$ and $i = 1, \ldots, d$, we must have $T = \infty$. Since $(x_1^i, \ldots, x_d^i)$ constant $d$-vector, the constant vector is $(0, \ldots, 0)$ by (1.2) so $(0, \ldots, 0)$ is UAS for (3.6).

An example of a system of equations handled by the Corollary is

$$\begin{align*}
\dot{x} &= -a_1 x(t - |y(t)|) \\
\dot{y} &= -a_2 y(t - |x(t)|).
\end{align*}$$

If $a_1 > 0$ and $a_2 > 0$, 0 is UAS. Although the lag functions $|x(t)|$ and $|y(t)|$ are not bounded by a finite $q$, by choosing $\beta > 0$ sufficiently small ($a_1 \beta < \frac{2}{3}$ and $a_2 \beta < \frac{1}{3}$) and restricting the domain of the equation to be $\{ |x| < \beta, |y| < \beta \}$, the conditions can be met. Note that the statement guarantees that solutions starting near 0 remain in this domain and are defined for all time.

Corollary 3.4 applies Theorem 1.1 to equations for which 0 is not a solution.

3.4. COROLLARY. Let \( G : [0, \infty) \times C^\alpha \to \mathbb{R} \) be continuous. Suppose that for some $\alpha > 0$, $\omega_1 q < M$ and

$$\alpha M(\theta - \psi) \geq G(t, \psi) - G(t, \theta) \quad \text{for all} \quad \theta, \psi \in C^\alpha, t \geq 0. \quad (3.8)$$

Then each noncontinuable solution $x(\cdot)$ of

$$\dot{x} = G(t, x(t)) \quad (3.9)$$

is defined on $[t_0, \infty)$ for some $t_0$, and for any other noncontinuable solution $y$, \( \lim_{t \to \infty} |x(t) - y(t)| = 0 \).

Proof. If 0 denotes the identically 0 function in $C^\alpha$, then for any $\phi$, $|G(t, \phi) - G(t, 0)| \leq \alpha \|\phi\|$. Hence

$$|G(t, \phi)| \leq \alpha \|\phi\| + |G(t, 0)|. \quad (3.10)$$

Using the Gronwall inequality on $\|x_t\|$ as in ordinary differential equations, (3.10) and Theorem 2.3 imply that each noncontinuable solution is defined until $+\infty$. Let $u(\cdot)$ be a noncontinuable solution on $[0, \infty)$ of (3.9). Define

$$F(t, \phi) = G(t, \phi + u_t) - G(t, u_t) \quad \text{for} \quad \phi \in C^\alpha.$$

$F$ satisfies (1.1) letting $\theta = \phi + u_t$ and $\psi = u_t$ in (3.8). For any noncontinuable solution $y$ of (3.9) at $t_0$, $z(t) = y(t) - u(t)$ satisfies (DDE) at $t_0$, so by Theorem 1.1

$$|z(t)| = |y(t) - u(t)| \to 0 \quad \text{as} \quad t \to \infty,$$

since by switching $\theta$ and $\psi$ in (3.8), $G(t, \theta) - G(t, \psi) \geq \alpha M(-|\theta - \psi|)$. Let $x$ be another noncontinuable solution of (3.9). Then $x(t) - u(t) \to 0$, so

$$x(t) - y(t) \to 0 \quad \text{as} \quad t \to \infty.$$
4. The Proof of the Main Result

In order to make condition (1.1) more intuitive, (and for later application) we first give two simple propositions. Then Lemma 4.3 is proved. The rest of the section completes the details of the proof of Theorem 1.1, which is proved in five steps.

4.1. Proposition. Assume (1.1) is satisfied for some $\alpha > 0$ and $\beta > 0$. Then

$$|F(t, \phi)| \leq \alpha \| \phi \| \quad \text{for} \quad \phi \in C^q_0.$$  \hspace{1cm} (4.1)

The proof is simple and is omitted.

4.2. Proposition. Assume (1.1) is satisfied for some $\alpha > 0$ and $\beta > 0$. Let $x(\cdot)$ be a solution on $[t_1, \tau]$ such that

$$x(t) \geq 0 \quad \text{for all} \quad t \in [t_1, \tau] \quad \text{and} \quad \dot{x}(\tau) > 0$$

or

$$x(t) \leq 0 \quad \text{for all} \quad t \in [t_1, \tau] \quad \text{and} \quad \dot{x}(\tau) < 0.$$

Then $\tau < t_1 + q$.

Proof. In the first case $\inf_{[t_1, \tau]} x(\cdot) \geq 0$. From (1.1)

$$0 > -\dot{x}(\tau) = -F(\tau, x) \geq -\alpha M(-x),$$

so $0 < \alpha M(-x)$ and $x(s) < 0$ for some $s \in [\tau - q, \tau]$. But $s < t_1$ so $\tau - q < t_1$. The second case is similar.

4.3. Lemma. Let $x(\cdot)$ be a solution on $[t_1 - \alpha^{-1}, T]$ for some $t_1$ and $T$, $0 < \alpha^{-1} < t_1 < T$, and assume $x(t_1) = 0$. Assume

$$0 < \rho_1 \overset{\text{def}}{=} \sup \{|x(t)| : t \in [t_1 - \alpha^{-1} - q, t_1]|.$$  \hspace{1cm} (4.2)

Assume $F$ satisfies (1.1) for some $\alpha$ such that $1 \leq \alpha q \leq \frac{\beta}{3}$. Then

$$|x(t)| \leq (\alpha q - \frac{1}{2}) \rho_1 \quad \text{for} \quad t \in [t_1, T].$$

Since $x(\cdot)$ is a solution on $[t_1 - \alpha^{-1}, T]$, $x$ is defined on at least $[t_1 - \alpha^{-1} - q, T]$.

Proof. Suppose the lemma is false. Then define

$$T_2 = \inf \{t > t_1 : \rho_1^{-1} |x(t)| > \alpha q - 1/2\}.$$

Choose $\epsilon > 0$ such that $|x(T_2 + \epsilon)| > |x(T_2)|$ and $|x(t)| \neq 0$ for
\( t \in [T_2, T_2 + \epsilon] \). Since \( |x(t)| \leq (\alpha g - \frac{1}{2}) p_1 \) for \( t \leq T_2 \), we may assume
\( t_1 = \sup\{ t < T_2 : x(t) = 0 \} \). If this were not so, we could redefine \( t_1 \). Since
\( |x(\cdot)| \) is increasing somewhere on \((T_2, T_2 + \epsilon)\), \( x(t) \) and \( \dot{x}(t) \) have the same
sign for some \( t \in (T_2, T_2 + \epsilon) \). Thus by Proposition 4.2,
\[
T_2 < t_1 + q
\]
(4.3)
We will assume \( x(T_2) > 0 \) since the argument for the case \( x(T_2) < 0 \) is
similar and is mainly a matter of changing signs. From Propositions 4.1 and 4.2,
\[
|\dot{x}(s)| = |F(s, x_s)| \leq \alpha \| x_s \| \leq \alpha p_1, \quad \text{for} \quad s \in (t_1 - \alpha^{-1}, T_2]. \quad (4.4)
\]
\[
|\dot{x}(s)| = |x(t_1) - x(s)| \leq \left| \int_{t_1}^{T_1} |\ddot{x}(t)| \, dt \right| \leq \alpha p_1 |t_1 - s|
\]
for \( s \in (t_1 - \alpha^{-1}, T_2] \) (4.5)
In fact
\[
|\dot{x}(s)| \leq \alpha p_1 |t_1 - s| \quad \text{for} \quad s \in [t_1 - q, T_2],
\]
since
\[
|x(s)| \leq p_1 \leq \alpha p_1 |t_1 - s| \quad \text{for} \quad s \in [t_1 - q, t_1 - \alpha^{-1}].
\]
For \( t_1 + s \in (t_1, T_2) \), an interval on which \( x(\cdot) \) is positive, the right-hand-side
of (1.1) implies
\[
\dot{x}(t_1 + s) = F(t_1 + s, x_{t_1+s}) \leq \alpha M(-x_{t_1+s})
\]
\[
= \alpha \sup\{ -x(t) : t_1 + s - q \leq t \leq t_1 \}
\]
\[
\leq \alpha \sup\{ x(t) : t_1 - s - q \leq t \leq t_1 \}
\]
\[
\leq \alpha \sup\{ \alpha p_1 (t_1 - t) : t_1 + s - q \leq t \leq t_1 \} = \alpha^2 p_1 (q - s).
\]
Hence, together with (4.4),
\[
\dot{x}(t_1 + s) \leq \inf\{ \alpha^2 (q - s) \} \rho_1 \quad \text{for} \quad t_1 + s \in [t_1, T_2]
\]
\[
= \left\{
\begin{array}{ll}
\alpha p_1 & \text{for} \quad 0 \leq s < q - \alpha^{-1},
\alpha^2 (q - s) \rho_1 & \text{for} \quad q - \alpha^{-1} \leq s \leq q.
\end{array}
\right.
\]
By (4.3) \( t_1 < T_2 < t_1 + q \), and \( x(T_2) \leq p_1 \), so writing \( q_1 = q - \alpha^{-1} \),
\[
x(T_2) = x(T_2) - x(t_1) = \int_0^{T_2 - t_1} \dot{x}(t_1 + s) \, ds \leq \int_0^{T_2 - t_1} \inf\{ \alpha^2 (q - s) \} \rho_1 \, ds
\]
\[
< \int_0^{q_1} \alpha p_1 \, ds + \int_{q_1}^{q} \rho_1 \alpha^2 (q - s) \, ds
\]
\[
= \left[ \alpha (q - \alpha^{-1}) + \frac{1}{2} \right] \rho_1 = \left[ \alpha q - \frac{1}{2} \right] \rho_1
\]
which is a contradiction since $x(T_2) = (\alpha q - 2^{-1}) \rho_1$ by definition of $T_2$. Hence no $T_2$ exists as assumed, so this lemma is proved.

Theorem 1.1 assumes that for some $\alpha$ sufficiently large (1.1) is satisfied. If (1.1) is satisfied by some $\alpha < q^{-1}$, it is also satisfied by letting $\alpha = q^{-1}$; that is, $\alpha$ has the role of an upper bound in (1.1) so if $q^{-1} > \alpha$, then $q^{-1}$ would also be an "upper bound" and we may let $\alpha = q^{-1}$. Hence we may assume without loss of generality that

$$\alpha q \geq 1.$$  \hfill (4.6)

**Proof of Theorem.** (i) Proof of uniform stability and of (1.4). The conditions of Proposition 4.1 are satisfied so (4.1) holds. Write $\rho = \|\phi\|$ where $\|\phi\| < 2\beta/5$. It suffices to prove that if $\rho > 0$ then for each solution $x(\cdot)$ with $x_{t_0} = \phi$, $|x(t)| \leq 5\rho/2$ for all $t \geq t_0$ in the domain of $x(\cdot)$. It follows that the same result holds for $\rho = 0$ and that 0 is stable. (4.1) implies $|F|$ is bounded (by $\alpha \beta$) on $[0, \infty) \times C_{\phi}$ so by Theorem 2.3 ii, it would follow that each such (noncontinuable) solution at $t_0$ is defined on $[t_0, \infty)$. We assume $\alpha q < \frac{1}{2}$.

Suppose (1.3) is false and there exists $T > t_0$ such that $|x(T)| > \frac{3}{2} \rho$. Let $T_1 = \inf\{t > t_0 : |x(t)| > \frac{3}{2} \rho\}$. Then $|x(T_1)| = \frac{3}{2} \rho$. It suffices to assume $x(T_1) > 0$. If instead $x(T_1) < 0$, this argument is similar and the changes are primarily a matter of changing signs. If $x(t) \neq 0$ for $t \in [t_0, T]$, define $t_1 = t_0$; otherwise, define $t_1 = \sup\{t < T_1 : x(t) = 0\}$ and then $x(t_1) = 0$. We may assume $T > T_1$ was chosen such that $x(t) > 0$ for $t \in [T_1, T]$ and so also in $(t_1, T)$ and such that $x(T) > 0$. Therefore by Proposition 4.2

$$t_1 < T_1 < T < t_1 + q.$$  \hfill (4.7)

Let $\rho_1 = \sup\{|x(t)| : t_0 - q \leq t \leq t_1\}$. The argument separates into two cases since $\rho < \rho_1$.

**Case 1.** Suppose $\rho < \rho_1$. Then $t_0 \neq t_1$ and $x(t_1) = 0$. To apply Lemma 4.3, we now show $t_0 \leq t_1 = \alpha^{-1}$. There exists $t_2 \in (t_0, t_1)$ such that $|x(t_2)| = \rho_1$. By Proposition 4.1, for $s \in (t_0, t_1)$

$$|x(s)| = |F(s, x_s)| \leq \alpha \|x_s\| \leq \alpha \rho_1,$$

$$\rho_1 = |x(t_2)| = |x(t_2) - x(t_2)| \leq \int_{t_2}^{t_1} |x(s)| \, ds \leq \alpha \rho_1 [t_1 - t_2],$$

or $1 \leq \alpha [t_1 - t_2]$, so $t_0 < t_2 \leq t_1 = \alpha^{-1}$ and the conditions of Lemma 4.3 are satisfied. Since $T \geq t_1$ and $\alpha q \leq \frac{1}{2}$, the lemma implies

$$|x(T)| \leq \rho_1(\alpha q - \frac{1}{2}) \leq \rho_1(\frac{1}{2} - \frac{1}{2}),$$

contradicting our assumption $x(T) > x(T_1) = \frac{3}{2} \rho \geq \rho_1$. Hence we cannot have $\rho < \rho_1$. 


Case 2. Since Case 1 is impossible, \( \rho_1 = \rho \) and \( t_1 \geq t_0 \). Since \( x(t) \geq 0 \) for \( t \in [t_1, T] \), \( x(t) \geq -\rho \) for \( t \in [t_0 - q, T] \), and by (1.1)

\[
\dot{x}(t) = F(t, x(t)) \leq -\alpha \inf x(t) \leq \alpha \rho
\]

for \( t \in [t_0, T] \). Either \( x(t_1) = 0 \) or \( t_0 = t_1 \) and \( x(t_1) \leq \rho \). Therefore

\[
\frac{5}{2} \rho = x(T_1) = x(t_1) + \int_{t_1}^{T_1} \dot{x}(t) \, dt \\
\leq \rho + \int_{t_1}^{T_1} \rho x(t) = \rho + \rho x(T_1 - t_1) \\
< \rho + \rho \alpha q \leq \rho \frac{5}{2}
\]

(from (4.7)),

which is a contradiction and both cases 1 and 2 are impossible. Therefore there exists no \( T \) satisfying \( x(T) \geq \frac{5}{2} \rho \) and (i) is verified. (ii) Proof that if \( \alpha q < \frac{3}{2} \), then \( \lim_{t \to \infty} x(t) \) exists.

There are two cases to consider. Suppose first that for some \( T \), \( x(t) \neq 0 \) for \( t > T \). By Proposition 4.2 \( x(t) \) and \( \dot{x}(t) \) have opposite sign for \( t \geq T + q \). Hence \( |x| \) is a monotonically decreasing function on \( (T + q, \infty) \) (bounded below by 0). Whether \( x(t) > 0 \) or \( < 0 \), \( x(t) \) constant as \( t \to \infty \). If no such \( T \) exists, the result is stronger. There exists a sequence \( \{\tau_i\}_{i=1}^{\infty} \) with \( \tau_i \to \infty \) as \( i \to \infty \) such that \( x(\tau_i) = 0 \) for each \( \tau_i \). The sequence may be chosen so that \( \tau_{i+1} > \tau_i + q \). Let \( \rho_i = \sup_{\tau_i \leq t \leq \tau_{i+1}} |x(t)| \) for \( i = 1, 2, \ldots \). By Lemma 4.3

\[
(\alpha q - \frac{3}{2}) \rho_i \geq \sup_{t \geq \tau_i} |x(t)| \geq \sup_{\tau_{i+1} \leq t \geq \tau_i} |x(t)| = \rho_{i+1} 
\]

By induction \( (\alpha q - \frac{3}{2}) \rho_1 \geq \rho_{n+1} \) so \( \rho_n \to 0 \) as \( n \to \infty \) since \( \alpha q - \frac{3}{2} < 1 \), and \( \lim_{t \to \infty} x(t) = 0 \), a constant so (ii) holds.

(iii) Proof of 1.3. From (ii) \( x(t) \to \gamma \) as \( t \to \infty \) for some \( \gamma \). It suffices to prove that \( \gamma = 0 \). Let \( \psi \in C^0 \) denote the constant function equal to \( \gamma \). From the proof of (ii), \( \alpha q < \frac{3}{2} \) implies \( \gamma = 0 \) or \( x(t) \to \) constant monotonically (for \( t \) large). In the latter case there exists a sequence \( t_n \to \infty \) such that \( \dot{x}(t_n) \to 0 \). Let \( \phi_n = x_{t_n} \). Since \( x(t) \to \gamma \), \( x(t) \to \psi \), that is,

\[
\sup_{t_0 \leq s \leq t} |x(s) - \gamma| \to 0
\]

as \( t \to \infty \), and \( \phi_n \to \psi \). Then \( \dot{x}(t_n) = F(t_n, \phi_n) \to 0 \) as \( n \to \infty \). From (1.2) must be 0, and \( x(t) \to 0 \) as \( t \to \infty \). Since whenever \( \|x(t_0)\| = \|\phi\| < \frac{3}{2} \beta \), \( x(t) \to 0 \) as \( t \to \infty \) and from (i) uniform stability, the zero solution is asymptotically stable.
(iv) Proof of (1.5) and (1.6). Let $V(t) = \sup_{t \leq s \leq t + 3q} |x(s)|$ for the solution $x$. It suffices to show $V(T) > |x(t)|$ for $t \geq T + 3q$ to show $V$ is non-increasing. From (4.6) $q \geq a^{-1}$. If $x(t_1) = 0$ for some $t_1 \in [T + 2q, T + 3q]$, then $T \leq t_1 - q - a^{-1} \leq t_1 \leq T + 3q$, so $V(T) \geq \rho_1$ where $\rho_1 = \sup_{t \in [t_1 - q - a^{-1}, t_1]} |x(t)|$. Then by Lemma 4.3, $|x(t)| \leq \rho_1 \leq V(T)$ for $t \geq t_1$ and we are finished. If there exists no $t_1 \in [T + 2q, T + 3q]$ such that $x(t_1) = 0$, then let $t_2$ be the first zero of $x(\cdot)$ greater than $T_1 + 3q$, or let $t_2 = \infty$ if no zero exists. By Proposition 4.2, $|x(t)|$ is monotonically decreasing for $t \in [T + 3q, t_2]$ so if $t_2 < \infty$, then $|x(t)|$ is monotonically decreasing for $t \in [T + 3q, t_2]$ and by Lemma 4.3

$$V(T) \geq \sup_{t_2 < s \leq t_2 + q} |x(s)| \geq |x(t)| \quad \text{for} \quad t \geq t_2$$

so in all cases $V$ is non-increasing. Since $x(t) \to 0$, $V(t) \to 0$ as $t \to \infty$ and the result is proved.

(v) Proof that 0 is UAS with attraction radius $2\beta/5$. If $x$ is a (noncontinuable) solution at $t$ and $|x_t| < 2\beta/5$, then by (1.4) $|x(s)| \leq 5|x_t|/2 < \beta$ for $s > t$ in the domain of $x(\cdot)$. By Proposition 4.1, $|\xi|$ is bounded on $[0, \infty) \times C_{\beta}$.

By Theorem 2.3 (ii), $x$ is a solution on $[t, \infty)$. Hence (i) and (ii) of Definition 2.4 are satisfied.

Suppose 0 is not UAS with attraction radius $2\beta/5$. There must exist $\gamma_1 \in (0, 2\beta/5)$ for which no $T(\gamma_1)$ exists; that is, there exists $\{s_n\}, \{t_n\}$, and a sequence $\{x^n\}$ of solutions of $[s_n, \infty)$, $n = 1, 2, \ldots$, such that $s_n > 0$, $|x^n_{s_n}| \leq \gamma_1$, but (2.1) fails for all $T$, i.e.,

$$T_n \to \infty \quad \text{and} \quad |x^n(s_n + T_n)| > \gamma_1/2. \quad (4.9)$$

Let $V_n(s) = \sup_{s \leq \sigma \leq s + 3q} |x^n(\sigma)|$ for $s \geq s_n$. Then from (1.4)

$$5\gamma_1/2 \geq V_n(s_n) \geq V_n(s_n + T_n) \geq |x^n(s_n + T_n)| > \gamma_1/2,$$

since by part (iv) of this theorem $V_n$ is monotonically decreasing. It follows that $V_n$ must be decreasing slowly somewhere on $[s_n, s_n + T_n]$ in the following sense. Write $\theta = \omega q - \frac{1}{2}$. By (4.6) $\theta \in [\frac{1}{4}, 1)$. There must exist sequences $\{s'_n\}$ and $\{T'_n\}$ such that $s_n < s'_n < s_n + T'_n < s_n + T_n$, and $T'_n \to \infty$, and

$$V_n(s'_n + T'_n) > \theta V_n(s'_n); \quad (4.10)$$

that is, $V_n$ decreases by less than a factor of $\theta$ on intervals of length $T'_n$. Since $T'_n \to \infty$, we may assume $T'_n > 3q$ for all $n$. We claim

$$x^n(t) \neq 0 \quad \text{for} \quad t \in [s'_n + 2q, s_n' + T'_n], \quad (4.11)$$
and

\[ |x^n| \text{ is monotonically decreasing on } [s_n' + 3q, s_n' + T_n']. \]  

(4.12)

Suppose (4.11) fails and for some \( z_n \in [s_n' + 2q, s_n' + T_n'], x^n(z_n) = 0 \). Let \( \rho_1 \) in (4.2) be \( \sup \{ |x^n(t)| : z_n - q - \alpha^{-1} < t \leq z_n \} \). By (4.6) \( 2q \geq q + \alpha^{-1} \), so since \( V(s_n) \geq |x^n(s_n + s)| \) for all \( s \geq 0 \) and

\[
V_n(s_n) \geq \sup_{\sigma > s_n} |x^n(\sigma)| \geq \sup_{\sigma > z_n, \sigma \rightarrow s_n} |x^n(\sigma)| \geq \rho_1 \\
\geq \sup_{s \geq 0} x^n(z_n + s) \theta^{-1} \quad \text{(by Lemma 4.3 and (4.2))} \\
\geq \theta^{-1} V_n(z_n) \quad \text{(by definition of } V_n) 
\]

which is \( \geq \theta^{-1} V_n(s_n' + T_n') \) (by monotonicity), contradicting (4.10). Hence \( x^n \) has no zero \( z_n \) and (4.11) is proved. Write \( I_n = [s_n' + 3q, s_n' + T_n'] \).

(4.11) and Proposition 4.2 imply that for \( t \in I_n \), \( x^n(t) x^n(t) \leq 0 \) and (4.12) is satisfied and

\[ |x^n(t)| \geq |x^n(s_n' + T_n')| > \theta \gamma_1, \quad \text{for } t \in I_n. \]

Since the length of \( I_n \) (i.e., \( T_n' \)) tends to \( \infty \), there exists \( \tau_n \) such that

\[ [\tau_n - 2q, \tau_n] \subset I_n \quad \text{and} \quad x^n(\tau_n - 2q) - x^n(\tau_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \]  

(4.13)

We may choose \( t_n \in [\tau_n - q, \tau_n] \) such that \( (d/dt) x^n(t_n) \rightarrow 0 \) as \( n \rightarrow \infty \). Since \( |x^n(t_n)| \in [\gamma_1/2, 5\gamma_1/2] \), some subsequence of \( \{x^n(t_n)\} \) converges to some constant \( c \neq 0 \). We may say without loss of generality \( x^n(t_n) \rightarrow c \). Since \( \dot{x}^n \) is monotonic, (4.13) implies \( x^n \) converges to the (nonzero) constant function in \( C^q \) equal to \( c \) as \( n \rightarrow \infty \), and \( (d/dt) x^n(t_n) = F(t_n, x^n(t_n)) \rightarrow 0 \) as \( n \rightarrow \infty \), contradicting (1.2) (letting \( \phi_n = x^n \)). This contradiction implies there exist no \( \gamma_1 \) and sequences of \( s_n \), \( T_n \), and \( x^n \) as assumed in (4.9). So \( 0 \) is UAS. It has already been shown that the attraction radius is \( 2\beta/5 \).

References


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