Perturbation Theorems for Ordinary Differential Equations

AARON STRAUSS AND JAMES A. YORKE*

Department of Mathematics,
University of Maryland, College Park, Maryland

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Consider the following systems of ordinary differential equations:

\[ x' = f(t, x), \]  
\[ x' = f(t, x) + g(t, x), \]  
\[ x' = f(t, x) + g(t, x) + h(t), \]

where \( f(t, x) \) is continuous, satisfies a Lipschitz condition on some semi-
cylinder, \( f(t, 0) = 0 \), and \( x = 0 \) is uniform asymptotically stable for \((N)\).
Let \( g(t, x) \) and \( h(t) \) be sufficiently smooth for local existence and uniqueness.

Consider the conditions

\((H_1)\): There exists \( r > 0 \) such that if \( |x| \leq r \), then \( |g(t, x)| \leq \gamma(t) \) for
all \( t \geq 0 \), where

\[ G(t) = \int_{t}^{t+1} \gamma(s) \, ds \to 0 \quad \text{as} \quad t \to \infty. \]  

\((H_2)\): There exists a continuous, nonincreasing function \( H(t) \) satisfying

\[ \lim_{t \to \infty} H(t) = 0 \]

such that \( |\int_{t_0}^{t} h(s) \, ds| \leq H(t_0) \) for every \( 0 \leq t_0 \leq t \leq t_0 + 1 \).

Then we prove: If \( g(t, x) \) satisfies \((H_1)\) and \( h(t) \) satisfies \((H_2)\), then there
exists \( T_0 \geq 0 \) and \( \delta_0 > 0 \) such that if \( t_0 \geq T_0 \) and \( |x_0| < \delta \), the solution
\( F(t, t_0, x_0) \) of \((P_1)\) approaches zero as \( t \to \infty \). In particular, if \( x = 0 \) is a
solution of \((P_1)\), then it is uniform asymptotically stable. Furthermore, if
\( g(t, x) \equiv 0 \) and \( h(t) \) does not satisfy \((H_2)\), then no solution of \((P_1)\) can approach
zero as \( t \to \infty \).

In the case \( f(t, x) = Ax \), where \( A \) is a constant matrix, the above results

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generalize theorems of Coddington and Levinson [1] and Brauer [2], since (1.1) holds if either \( y(t) \to 0 \) as \( t \to \infty \) or \( \int_0^\infty y(t) \, dt < \infty \). The proof for the linear case is elementary and can be given in a first course in differential equations; therefore, we include it in Section 3.

We give two proofs of the general result. A slight extension of the above is proved in Section 4 with Lyapunov functions. The proof in Section 5 is more direct, does not require Lyapunov's second method, and exhibits quite clearly the need for the asymptotic stability to be uniform. Example 4.5 shows that a vector function \( h(t) \) can satisfy \((H_3)\) even though \( |h(t)| \to \infty \) as \( t \to \infty \). Needless to say, all of the general results of Sections 4 and 5 apply to the special case \( f(t, x) = Ax \) considered in Section 3.

Lyapunov's second method has been employed before to obtain perturbation theorems. One of the best-known results [3, Section 19] is that if \( f \in \mathcal{C}_0 \) and \( x = 0 \) is U.A.S. for \((N)\), then there exists a continuous function \( \eta(x) \), with \( \eta(0) = 0 \), such that if \( \|g(t, x)\| \leq \eta(x) \), then \( x = 0 \) is U.A.S. for \((P)\). This theorem has wide application, and it is probably close to the best possible result for perturbation terms that are independent of \( t \). The difficulty is that it may be hard to determine \( \eta(x) \) because \( \eta \) depends on the Lyapunov function associated with \((N)\). In our theorems, a perturbation term need only satisfy a requirement which is independent of the original system \((N)\).

Krasovskii [3, Section 24] has a theorem in the general spirit of Theorem 4.1 in which he considers perturbation terms that are "bounded in the mean." He is able, however, to conclude only that \( x = 0 \) is stable for the perturbed system. Also, a result similar to the above for \((P)\) was announced by LaSalle and Rath [5]. Instead of (1.1), they use the equivalent (but perhaps a bit harder to verify) condition

\[
\limsup_{t \to \infty} \frac{1}{1 + a} \int_t^{t + a} \gamma(s) \, ds = 0
\]

for some interesting theorems on "eventual stability." Actually, the condition (1.1) seems to have been used only recently—Coppel [4] uses it to obtain several interesting boundedness results for perturbed linear systems.

2. Preliminaries

In \((N)\), \( x \) and \( f \) are vectors in \( E_n \) with \( \|x\| = |x_1| + \cdots + |x_n| \), \( t \) is real, and \( f(t, 0) = 0 \) for all \( t \geq 0 \). We adopt the following convention: every differential equation which we consider shall have a right-hand side which is continuous and sufficiently smooth on

\[ D_M = \{(t, x) : t \geq 0, \quad |x| < M, \quad M > 0\} \]
for the uniqueness of all solutions. For \((t_0, x_0) \in D_M\) we denote by \(F(t, t_0, x_0)\) that solution (of the equation being considered) for which \(F(t_0, t_0, x_0) = x_0\).

**Definition 2.1.** \(C_0\) denotes the class of functions having uniform Lipschitz constants on \(D_M\); \(C_m\) the class having continuous partial derivatives of orders \(k = 1, 2, \ldots, m\); and \(C_\infty\) the class of functions having continuous and bounded partial derivatives of every order on \(D_M\).

**Definition 2.2.** \(\mathcal{X}\) is the class of continuous, strictly increasing functions \(\rho\) on \([0, M)\) such that \(\rho(0) = 0\).

**Definition 2.3.** Let \(V(t, x)\) be real valued on \(D_M\). Then \(V\) is positive definite if there exists \(\rho \in \mathcal{X}\) such that \(V(t, x) \geq \rho(|x|)\) on \(D_M\). Also, \(V\) is negative definite if \(-V\) is positive definite.

**Definition 2.4.** \(V(t, x)\) is a Lyapunov function for \((N)\) on \(D_M\) if

(i) \(V(t, x)\) is positive definite and \(C_1\) on \(D_M\),
(ii) \(V(t, 0) = 0\) for all \(t \geq 0\),
(iii) \(\dot{V}_{(N)}(t, x) = \partial/\partial t V(t, x) + \nabla V(t, x) \cdot f(t, x) \leq 0\) on \(D_M\), where

\[
\nabla V = \left( \frac{\partial}{\partial x_1} V, \ldots, \frac{\partial}{\partial x_n} V \right).
\]

**Definition 2.5.** \(V(t, x)\) has an infinitesimal upper bound on \(D_M\) if there exists \(\mu \in \mathcal{X}\) such that \(V(t, x) \leq \mu(|x|)\) on \(D_M\).

**Definition 2.6.** Call the solution \(x = 0\) of \((N)\)

(2.1) stable if for every \(\epsilon > 0\) and every \(t_0 \geq 0\), there exists \(\delta(\epsilon, t_0) > 0\) such that \(|x_0| < \delta\) and \(t \geq t_0\) imply \(|F(t, t_0, x_0)| < \epsilon\).

(2.2) uniformly stable if (2.1) holds with \(\delta\) independent of \(t_0\).

(2.3) asymptotically stable if (2.1) holds and if there exists \(\delta_0(t_0) > 0\) such that or every \(\eta > 0, t_0 \geq 0\), and \(|x_0| < \delta_0\), there exists \(T(\eta, t_0, x_0) \geq 0\) such that \(t \geq t_0 + T\) implies \(|F(t, t_0, x_0)| < \eta\).

(2.4) equiasymptotically stable (E.A.S.) if (2.3) holds with \(T\) independent of \(x_0\).

(2.5) uniform-asymptotically stable (U.A.S.) if (2.2) and (2.3) hold with \(\delta_0\) independent of \(t_0\) and \(T\) independent of \(t_0\) and \(x_0\).

**Theorem 2.7** (Massera [5]). If, for \((N)\), \(f \in \mathcal{C}_0\) on \(D_M\) and \(x = 0\) is U.A.S., then there exists on \(D_M\) a Lyapunov function \(V(t, x)\) for \((N)\) such that \(V\) has an infinitesimal upper bound, \(\dot{V}_{(N)}\) is negative definite, and \(V \in \mathcal{C}_\infty\).
3. Perturbed Linear Systems

The following is proved in [1, Chapter 13]:

**Theorem 3.1.** In the system

\[ x' = Ax + \psi(t, x) + g(t, x) \]  

let \( \psi \) and \( g \) be continuous on \( D_M \). For small \( |x| \), let \( g(t, x) \to 0 \) as \( t \to \infty \) uniformly in \( x \). Let the characteristic roots of \( A \) have negative real parts. Given any \( \epsilon > 0 \), let there exist \( \delta_\epsilon \) and \( T_\epsilon \) so that \( |\psi(t, x)| \leq \epsilon |x| \) for \( |x| \leq \delta_\epsilon \) and \( t \geq T_\epsilon \). Then there exists \( T_0 \) such that any solution \( F(t) \) of (L) satisfies \( |F(t)| \to 0 \) as \( t \to \infty \) if \( |F(T_0)| \) is small enough.

Brauer [2] has shown that the conclusion of Theorem 3.1 is valid for the system

\[ x' = Ax + \psi(t, x) + g(t, x) + h(t, x), \]

where \( A, \psi, \) and \( g \) are as in Theorem 3.1, \( h(t, x) \) is continuous on \( D_M \) and \( |h(t, x)| \leq \lambda(t) |x| \), where \( \int_0^\infty \lambda(t) \, dt < \infty \).

We shall now consider (L) with a condition on \( g(t, x) \) sufficiently general to include Brauer's result as a special case.

**Theorem 3.2.** Consider (L) where \( A \) and \( \psi \) are as in Theorem 3.1, and \( g \) satisfies (H1). Then there exist \( T_0 \geq 0 \) and \( \delta > 0 \) such that for every \( \tau_0 \geq T_0 \) and \( x_0 \) satisfying \( |x_0| < \delta \), the solution \( F(t, \tau_0, x_0) \) of (L) satisfies \( |F(t, \tau_0, x_0)| \to 0 \) as \( t \to \infty \). In particular, if \( x = 0 \) is a solution of (L), then it is asymptotically stable.

Before proving this theorem, we discuss the condition (H1). It is easily seen that (1.1) holds if either \( \gamma(t) \to 0 \) as \( t \to \infty \) or \( \int_0^\infty \gamma(t) \, dt < \infty \); that is, Theorem 3.2 includes Brauer's result. The following example exhibits the generality of (1.1).

**Example 3.3.** Define \( \gamma(t) \) on \([0, \infty)\) as follows: for each positive integer \( n \),

\[ \gamma(3n) = 1, \quad \gamma(t) = 0 \text{ on } [3n - n^{-1}, 3(n + 1) - (n + 1)^{-1}], \quad \gamma(t) = 0 \text{ on } [0, 2], \]

and \( \gamma \) is linear elsewhere. Then \( \gamma(t) \to 0 \) as \( t \to \infty \) and \( \int_0^\infty \gamma(t) \, dt = \infty \) because

\[ \int_0^{3n+1} \gamma(t) \, dt = \sum_{m=1}^{n} \frac{1}{m}. \]

However, given any \( t \geq 1 \), we have \( 3n - 2 \leq t \leq 3n + 1 \) for some \( n \), hence \( \int_t^{t+1} \gamma(s) \, ds \leq n^{-1} \), proving \( \gamma \) satisfies (1.1).

It is also easy to see that (1.1) is equivalent to the condition that
perturbation theorems

\[ \int_{t_0}^{t_\infty} y(s) \, ds \to 0 \text{ as } t \to \infty \] for every \( \alpha > 0 \); we use \( \alpha = 1 \) for convenience.

We prove Theorem 3.2 with the aid of four lemmas. Lemma 3.6 is analogous to Lemma 1 of [2] and Lemma 3.7 is an inequality of the Gronwall type used in [2].

**Lemma 3.4.**

\[ \int_{t_0}^{t} G(s) \, ds = \int_{t_0}^{t} \left[ \int_{s}^{t} y(u) \, du \right] ds \leq \int_{t_0}^{t} \left[ \int_{t_0}^{t} y(u) \, du \right] ds = \int_{t_0}^{t} y(u) \, du. \]

**Proof.**

\[ \int_{t_0}^{t} G(s) \, ds = \int_{t_0}^{t} \left[ \int_{s}^{t} y(u) \, du \right] ds \leq \int_{t_0}^{t} \left[ \int_{t_0}^{t} y(u) \, du \right] ds = \int_{t_0}^{t} y(u) \, du. \]

**Lemma 3.5.**

\[ \int_{t_0}^{t} e^{\sigma s} y(s) \, ds \leq \int_{t_0}^{t} e^{(\sigma + 1) G(s)} \, ds \quad \text{for all } \sigma > 0, t \geq t_0 \geq 1. \]

**Proof.** By using Lemma 3.4,

\[ \int_{t_0}^{t} e^{\sigma s} y(s) \, ds \leq \int_{t_0}^{t} e^{(\sigma + 1) G(s)} \, ds \leq \int_{t_0}^{t} e^{(\sigma + 1) G(s)} \, ds. \]

**Lemma 3.6.**

\[ \lim_{t \to \infty} e^{-\sigma t} \int_{t_0}^{t} e^{\sigma s} y(s) \, ds = 0 \quad \text{for all } \sigma > 0. \]

**Proof.** Using Lemma 3.5 and then L'Hospital's rule on \( \int_{t_0}^{t} e^{(\sigma + 1) G(s)} \, ds \) gives the result.

**Lemma 3.7.** Let \( r(t) \) and \( p(t) \) be continuous for \( t \geq t_0 \), let \( c \geq 0, k \geq 0, \) and let

\[ r(t) \leq c + \int_{t_0}^{t} [kr(s) + p(s)] \, ds. \]

Then

\[ r(t) \leq ce^{k(t-t_0)} + \int_{t_0}^{t} p(s)e^{k(t-s)} \, ds. \]

**Proof.** Let

\[ R(t) = c + \int_{t_0}^{t} [kr(s) + p(s)] \, ds. \]
Then $R' - kR \leq p$. The result follows by integrating both sides from $t_0$ to $t$ and solving for $R(t)$.

**Proof of Theorem 3.2 (using Brauer’s techniques).** Let $K \geq 1$ and $\sigma > 0$ such that $|e^{\sigma t}| \leq Ke^{-\sigma t}$ for all $t \geq 0$. Let $0 < \epsilon < \min(\sigma K^{-1}, \tau)$. From the hypothesis on $\psi$, choose $T_\epsilon$ and $\delta_\epsilon$ so that $\delta_\epsilon \leq \epsilon$ and $T_\epsilon \geq 1$. Choose $T_0 \geq T_\epsilon$ so large that $t \geq T_0$ implies

$$K \int_1^t e^{-(\sigma - K \epsilon)(t-s)} \gamma(s) \, ds < \frac{\delta_\epsilon}{2}.$$ 

This is possible by Lemma 3.6. Finally, let $\delta = \delta [2K]^{-1}$ and consider any $t_0 \geq T_0$ and $x_0$ satisfying $|x_0| < \delta$. Then for as long as $|F(t, t_0, x_0)| < \delta$, we have

$$F(t) = e^{At-t_0}x_0 + \int_{t_0}^t e^{A(t-s)}\psi(s, F(s)) \, ds + \int_{t_0}^t e^{A(t-s)}g(s, F(s)) \, ds,$$

from which

$$|F(t)| \leq K\delta e^{-(t-t_0)} + \int_{t_0}^t \left[ eK e^{-\sigma(t-s)} |F(s)| + Ke^{-\sigma(t-s)} \gamma(s) \right] \, ds$$

and

$$|F(t)| e^{\sigma t} \leq K\delta e^{\sigma t_0} + \int_{t_0}^t \left[ eK |F(s)| e^{\sigma s} + Ke^{\sigma s} \gamma(s) \right] \, ds.$$ 

Lemma 3.7 applied to $r(t) = |F(t)| e^{\sigma t}$ yields

$$|F(t)| e^{\sigma t} \leq K\delta e^{\sigma t-t_0} + \int_{t_0}^t Ke^{\sigma s} \gamma(s) e^{K(t-s)} \, ds,$$

so that

$$|F(t)| \leq K\delta e^{-(\sigma - K \epsilon)(t-t_0)} + K \int_{t_0}^t e^{-(\sigma - K \epsilon)(t-s)} \gamma(s) \, ds,$$

hence

$$|F(t)| \leq K\delta + K \int_1^t e^{-(\sigma - K \epsilon)(t-s)} \gamma(s) \, ds < K\delta + \frac{\delta_\epsilon}{2} = \delta_\epsilon.$$ 

Thus, the inequality $|F(t, t_0, x_0)| < \delta_\epsilon$ holds on $[t_0, \infty)$ which implies that (3.1) holds on $[t_0, \infty)$, hence $|F(t, t_0, x_0)| \to 0$ as $t \to \infty$. Since $\delta_\epsilon \leq \epsilon$, we have $|F(t, t_0, x_0)| < \epsilon$ on $[t_0, \infty)$, which gives the asymptotic stability of $x = 0$ when $g(t, 0) = 0$, completing the proof.

Our final example shows that in a weak sense, condition $(H_0)$ is necessary in order that solutions tend to zero.
Example 3.8. Consider the first order equation

\[ x' = -ax + \gamma(t), \quad (3.2) \]

where \( a > 0, \gamma(t) \geq 0 \) (this condition is removed in Theorem 4.7), and \( \gamma \) does not satisfy (1.1). Then there exists \( \alpha > 0 \) and a sequence \( t_n \to \infty \) as \( n \to \infty \) such that \( \int_{t_n}^{t_n+1} \gamma(t) \, dt \geq \alpha \) for every \( n \). Then for every choice of \( x_0 \geq 0 \) and \( t_0 \geq 0 \),

\[
F(t, t_0, x_0) = e^{-a(t-t_0)}x_0 + e^{-at} \int_{t_0}^{t} e^{as}\gamma(s) \, ds \geq e^{-at} \int_{t_0}^{t} e^{as}\gamma(s) \, ds.
\]

Hence for all large \( n \),

\[
F(t_n + 1, t_0, x_0) \geq e^{-a(t_n+1)} \int_{t_0}^{t_n+1} e^{as}\gamma(s) \, ds \geq e^{-a\alpha}.
\]

Thus \( F(t, t_0, x_0) \to 0 \) as \( t \to \infty \), and the conclusion of Theorem 3.2 does not hold for (3.2).

4. Perturbed Nonlinear Systems

We start with a slightly more general theorem than that mentioned in the introduction.

**Theorem 4.1.** Let \( x = 0 \) be U.A.S. and \( f \in C_0 \) on \( D_M \) for (N). Let \( g(t, x) \) satisfy

\[(H_g): \text{There exists } r > 0 \text{ such that for every } b, 0 < b < r, \text{ there exist } \tau_b > 0 \text{ and a function } \gamma_b(t) \text{ continuous on } [\tau_b, \infty) \text{ such that } |g(t, x)| \leq \gamma_b(t) \text{ for all } b \leq |x| \leq r \text{ and } t \geq \tau_b, \text{ where}
\]

\[ G_b(t) = \int_{t}^{t+1} \gamma_b(s) \, ds \to 0 \quad \text{as} \quad t \to \infty. \]

Then there exist \( T_0 > 0 \) and \( \delta_0 > 0 \) such that if \( t_0 \geq T_0 \) and \( |x_0| < \delta_0 \), then the solution \( F(t, t_0, x_0) \) of (P) satisfies \( |F(t, t_0, x_0)| \to 0 \) as \( t \to \infty \). In particular, if \( g(t, 0) = 0 \), then \( x = 0 \) is U.A.S. for (P).

**Proof.** By Theorem 2.7, there exists a Lyapunov function \( V \) for (N) on \( D_M \) satisfying

\[
\rho(|x|) \leq V(t, x) \leq \mu(|x|), \quad (4.1)
\]

\[
\dot{V}_{(N)}(t, x) \leq -\sigma(|x|), \quad (4.2)
\]

\[
|\nabla V(t, x)| \leq K, \quad (4.3)
\]
where \( \rho, \mu, \) and \( \sigma \) belong to \( \mathcal{K} \) and \( K \) is a positive constant. Now for as long as a solution \( F(t) \) of (P) exists,

\[
\frac{d}{dt} V(t, F(t)) = \frac{\partial}{\partial t} V(t, F(t)) + \nabla V(t, F(t)) \cdot [f(t, F(t)) + g(t, F(t))]
\]

\[
= \dot{V}(t, F(t)) + \nabla V(t, F(t)) \cdot g(t, F(t)).
\]

By integrating over any interval on which \( F(t) \) exists,

\[
(t, F(t)) - V(t_0, F(t_0)) = \int_{t_0}^{t} \dot{V}(s, F(s)) \, ds + \int_{t_0}^{t} \nabla V(s, F(s)) \cdot g(s, F(s)) \, ds
\]

\[
\leq - \int_{t_0}^{t} \sigma(|F(s)|) \, ds + K \int_{t_0}^{t} |g(s, F(s))| \, ds.
\]

Thus if \( 0 < b \leq |F(s)| < r \) between \( t_0 \) and \( t \),

\[
V(t, F(t)) \leq - \int_{t_0}^{t} \sigma(|F(s)|) \, ds + K \int_{t_0}^{t} G_b(s) \, ds + V(t_0, F(t_0)).
\]

If \( t \geq t_0 \geq 1 \), then Lemma 3.4 yields

\[
V(t, F(t)) \leq - \sigma(b)(t - t_0) + KQ_b(t_0)(b - 1) + V(t_0, F(t_0)).
\]

Define

\[
Q_b(t) = \sup\{G_b(s) : t - 1 \leq s < \infty\}.
\]

Then \( Q_b(t) \searrow 0 \) as \( t \to \infty \) and

\[
V(t, F(t)) \leq - \sigma(b)(t - t_0) + KQ_b(t_0)(b - 1) + V(t_0, F(t_0)),
\]

hence

\[
V(t, F(t)) \leq [KQ_b(t_0) - \sigma(b)](t - t_0) + KQ_b(t_0) + \mu(|F(t)|).
\]

Let \( 0 < \epsilon \leq r \). Choose \( \delta = \delta(\epsilon) \), \( 0 < \delta < \epsilon \), so that \( 2\mu(\delta) < \rho(\epsilon) \). Then choose \( T_1(\epsilon) \geq \tau_\delta + 1 \) so that

\[
2KQ_b(T_1) < \min\{\sigma(\delta), \rho(\epsilon)\}.
\]

Let \( |x_0| < \delta \) and \( t_0 \geq T_1 \). Then we claim

\[
|F(t, t_0, x_0)| < \epsilon \text{ on } [t_0, \infty).
\]

Suppose not. Let \( T_3 \) be the first point such that \( |F(T_3)| = \epsilon \) and let
$T_2 < T_3$ be the last point such that $|F(T_2)| = \delta$. Then $\delta \leq |F(t)| \leq \epsilon \leq r$ on $[T_2, T_3]$, hence by (4.5),

$$
\rho(\epsilon) \leq V(T_2, F(T_3)) \leq [KQ_0(T_3) - \sigma(\delta)](T_3 - T_2) + KQ_0(T_2) + \mu(|F(T_2)|)
$$

$$
\leq KQ_0(T_1) + \mu(\delta)
$$

$$
< \frac{1}{2}\rho(\epsilon) + \frac{1}{2}\rho(\epsilon) = \rho(\epsilon),
$$

a contradiction, proving (4.7). This proves the uniform stability of $x = 0$ for the case $g(t, 0) = 0$. For the rest of the proof choose $\delta_0 = \delta(r)$ and $T_0 = T_1(r)$. Fix $t_0 \geq T_0$ and $|x_0| < \delta_0$. Then (4.7) implies that $|F(t, t_0, x_0)| < r$ on $[t_0, \infty)$.

Let $0 < \eta < r$. Choose $\delta(\eta)$ and $T_1(\eta)$ as before so that (4.6) holds. Choose

$$
T = [\sigma(\delta)T_1(\eta) + 2KQ_0(1) + 2\mu(r)|\sigma(\delta)|^{-1}] > T_1(\eta)
$$

and it is clear that $T$ depends only on $\eta$, not on $t_0$ or $x_0$. We now claim

$$
|F(t_1, t_0, x_0)| < \delta \quad \text{for some } t_1 \text{ in } [t_0 + T_1, t_0 + T]. \quad (4.8)
$$

Suppose not. Then $|F(t, t_0, x_0)| \geq \delta$ on $[t_0 + T_1, t_0 + T]$. Let $y_0 = F(t_0 + T_1, t_0, x_0)$. Then

$$
0 < \rho(\delta) \leq \rho(|F(t_0 + T, t_0 + T_1, y_0)|)
$$

$$
\leq V(t_0 + T, F(t_0 + T))
$$

$$
\leq [KQ_0(t_0 + T_1) - \sigma(\delta)](T - T_1) + KQ_0(t_0 + T_2) + \mu(|y_0|)
$$

$$
< -\frac{1}{2}\sigma(\delta)(T - T_1) + KQ_0(1) + \mu(r) = 0,
$$

a contradiction, proving (4.8). Thus by (4.7)

$$
|F(t, t_1, F(t_1, t_0, x_0))| < \eta \quad \text{on } [t_1, \infty)
$$

because $t_1 \geq t_0 + T_1 \geq T_1$ and $|F(t_1, t_0, x_0)| < \delta$. Hence, a fortiori, $|F(t, t_0, x_0)| < \eta$ for $t \geq t_0 + T$. Since $\eta$ is arbitrary $|F(t, t_0, x_0)| \to 0$ as $t \to \infty$. Since $T$ depends only on $\eta$ and $\delta$ depends only on $\epsilon$, $x = 0$ is U.A.S. if $g(t, 0) = 0$, and the proof is complete.

Clearly, any function satisfying $(H_1)$ satisfies $(H_2)$. The converse is not true as the following example shows.

**Example 4.2.** Define $g(t, x) = t[t^2x + 1]^{-1}$. Then $g$ does not satisfy $(H_1)$ because $g(t, 0) = t$, but $g$ does satisfy $(H_2)$ with

$$
\tau_0 = b^{-1} + 1, \quad \gamma_0(t) = t[b^{-1} - 1]^{-1},
$$

and

$$
G_0(t) = [2b] \log[(t + 1)b^{-1} - 1][tb^{-1} - 1]^{-1}.
$$
The next example shows that the hypothesis "$x = 0$ is U.A.S." cannot be weakened to the condition "$x = 0$ is uniformly stable and E.A.S."

**Example 4.3.** Consider the first order equations

$$x' = -(t + 1)x,$$  \hspace{1cm} (4.9)

$$x' = -(t + 1)x + 2(t + 1)^{-1}x = (t + 1)^{-1}x. \hspace{1cm} (4.10)$$

Then $x = 0$ is uniformly stable and E.A.S. for (4.9), the perturbation term $2(t + 1)^{-1}x$ in (4.10) satisfies $(H_1)$, but all nontrivial solutions of (4.10) are unbounded.

The condition $f \in \mathcal{C}_0$ on $D_M$ need not hold, however. The existence of a Lyapunov function satisfying (4.1), (4.2), and (4.3) is sufficient for the conclusion of Theorem 4.1, and such a function may exist even though $f \notin \mathcal{C}_0$, as can be seen below.

**Example 4.4.** Consider the first-order equation

$$x' = -(t + 1)x,$$  \hspace{1cm} (4.11)

whose right-hand side is not in $\mathcal{C}_0$, yet the function $V(t, x) = x^2$ satisfies (4.1), (4.2), and (4.3). Thus (4.11) can be perturbed by a function satisfying $(H_2)$, although Theorem 4.1 as stated does not apply to (4.11).

We now consider the possibility of integrating the perturbation term before obtaining bounds on its norm. Specifically, consider the condition $(H_a)$: Let $h(t)$ be continuous on $[0, \infty)$ and let $H(t)$ be a continuous, nonincreasing function with $H(t) \to 0$ as $t \to \infty$ such that

$$\left| \int_{t_0}^t h(s) \, ds \right| \leq H(t_0) \quad \text{for every} \quad 0 \leq t_0 \leq t \leq t_0 + 1.$$  

Then any function $h(t)$ for which $\int_{t}^{t+1} |h(s)| \, ds \to 0$ as $t \to \infty$ satisfies $(H_3)$, so that perturbation terms independent of $x$ which satisfy $(H_1)$ also satisfy $(H_3)$. Example 4.5 shows the generality of $(H_3)$.

**Example 4.5.** Let $h(t) = (t \sin t^3, t \cos t^3)$. Then $h(t)$ does not satisfy $(H_1)$ because $|h(t)| \to \infty$ and hence $\int_t^{t+1} |h(s)| \, ds \to \infty$, as $t \to \infty$. But $h(t)$ satisfies $(H_3)$, because, for example,

$$\left| \int_{t_0}^t s \cos s^3 \, ds \right| = \left| \int_{t_0}^t \left[ s \cos s^3 - \frac{1}{3s^2} \sin 3s^3 \right] \, ds \right| + \int_{t_0}^t \frac{1}{3s^2} \sin 3s^3 \, ds \leq \left| \left[ \frac{1}{3s^3} \right]_{t_0}^t \right| + \int_{t_0}^t \frac{1}{3s^2} \, ds \leq \frac{1}{3t} + \frac{1}{3t_0} - \frac{1}{3t} + \frac{1}{3t_0} = \frac{2}{3t_0}.$$
Theorem 4.6. Let $x = 0$ be U.A.S. and $f \in \mathcal{C}_0$ on $D_m$ for $(N)$. Let $g$ satisfy $(H_2)$ and $h$ satisfy $(H_3)$. Then there exist $T_0 > 0$ and $\delta_0 > 0$ such that if $t_0 \geq T_0$ and $|x_0| < \delta_0$, the solution $F(t, t_0, x_0)$ of

$$x' = f(t, x) + g(t, x) + h(t)$$  \hspace{1cm} (P_1)$$

satisfies $|F(t, t_0, x_0)| \to 0$ as $t \to \infty$.

Proof. In deriving the analog of (4.4) for the system $(P_1)$, we are led to

$$V(t, F(t)) \leq - \int_{t_0}^t \sigma(|F(s)|) ds + f \int_{t_0}^t G_0(s) ds$$

$$+ \int_{t_0}^t \nabla V(s, F(s)) \cdot h(s) ds + V(t_0, F(t_0)). \hspace{1cm} (4.12)$$

Thus, we must estimate $|\int_{t_0}^t \nabla V(s, F(s)) \cdot h(s) ds|$.

Let $J(t) = \int_{t_0}^t h(s) ds$. Then

$$\int_{t_0}^t \nabla V(s, F(s)) \cdot h(s) ds$$

$$= \int_{t_0}^t \nabla V(s, F(s)) \cdot F'(s) ds + \int_{t_0}^t \frac{\partial}{\partial s} V(s, F(s)) ds - \int_{t_0}^t \frac{\partial}{\partial s} V(s, F(s)) ds$$

$$- \int_{t_0}^t \nabla V(s, F(s)) \cdot [f(s, F(s)) + g(s, F(s))] ds$$

$$+ \int_{t_0}^t \nabla V(s, F(s) - J(s)) \cdot [f(s, F(s)) + g(s, F(s))] ds$$

$$- \int_{t_0}^t \nabla V(s, F(s) - J(s)) \cdot [f(s, F(s)) + g(s, F(s))] ds$$

$$+ \int_{t_0}^t \frac{\partial}{\partial s} V(s, F(s) - J(s)) ds - \int_{t_0}^t \frac{\partial}{\partial s} V(s, F(s) - J(s)) ds$$

$$= \{V(t, F(t)) - V(t, F(t) - J(t)) \}$$

$$+ \int_{t_0}^t \left\{ \frac{\partial}{\partial s} V(s, F(s) - J(s)) - \frac{\partial}{\partial s} V(s, F(s)) \right\} ds$$

$$+ \int_{t_0}^t \{\nabla V(s, F(s) - J(s)) - \nabla V(s, F(s))\} \cdot \{f(s, F(s)) + g(s, F(s))\} ds.$$

We shall now use two facts which were not needed before; namely, $f$ is bounded on $D_m$ and $V$ satisfies

$$|V(t, x) - V(t, y)| \leq A |x - y|,$$

$$|\nabla V(t, x) - \nabla V(t, y)| \leq A |x - y|.$$
and
\[ \left| \frac{\partial}{\partial t} V(t, x) - \frac{\partial}{\partial t} V(t, y) \right| \leq A |x - y| \]
on \(D_M\) for some constant \(A > 0\). This latter fact holds because the first and second partials of \(V\) are bounded on \(D_M\) (by Theorem 2.7), while the former holds because \(f \in \mathcal{C}_0\) and \(f(t, 0) = 0\).

Let \(|f(t, x)| \leq B\) on \(D_M\). Temporarily, let \(0 < t - t_0 < 1\). Then \(|f(t)| \leq H(t_0)\). Thus
\[
\left| \int_{t_0}^t \nabla V(s, F(s)) \cdot h(s) \, ds \right|
\leq A |f(t)| + A \int_{t_0}^t |f(s)| \, ds + AB \int_{t_0}^t |f(s)| \, ds + A \int_{t_0}^t |f(s)| \gamma_0(s) \, ds
\leq [2A + AB] H(t_0) + AH(t_0) \int_{t_0-1}^t G_0(s) \, ds
= [2A + AB] H(t_0) + AH(t_0) + AH(t_0)Q_0(t_0)(t - t_0 + 1)
\leq [2A + AB + 2AQ_0(t_0)] H(t_0).
\]
Thus for arbitrary \(t > t_0\),
\[
\left| \int_{t_0}^t \nabla V(s, F(s)) \cdot h(s) \, ds \right|
\leq \left| \int_{t_0}^{t_0+1} \nabla V(s, F(s)) \cdot h(s) \, ds \right| + \cdots + \left| \int_{t_0+m}^t \nabla V(s, F(s)) \cdot h(s) \, ds \right|
\leq A[2 + B + 2Q_0(t_0)] H(t_0) + \cdots + A[2 + B + 2Q_0(t_0 + m)] H(t_0 + m)
\leq A[2 + B + 2Q_0(t_0)] H(t_0) + \cdots + A[2 + B + 2Q_0(t_0)] H(t_0)
\leq A[2 + B + 2Q_0(t_0)] H(t_0)[t - t_0 + 1].
\]
The proof now proceeds as the proof of Theorem 4.1.

If \(g(t, x) \equiv 0\) in \((P_1)\), then the condition \((H_3)\) on \(h(t)\) becomes necessary for the solutions to tend to zero.

**Theorem 4.7.** Let \(x = 0\) be U.A.S. and \(f \in \mathcal{C}_0\) on \(D_M\) for \((N)\). Then the conclusion of Theorem 4.6 holds for the system
\[ x' = f(t, x) + h(t) \quad (P_2) \]
if and only if \(h(t)\) satisfies \((H_3)\).

**Remark.** In \((P_2)\), \(h(t)\) is a vector; compare this result with Example 3.8.
Proof. Let $h$ satisfy $(H_3)$. Then the conclusion of Theorem 4.6 holds for $(P_2)$ by Theorem 4.6.

Conversely, suppose that $h$ does not satisfy $(H_3)$. Then there exist $\alpha > 0$ and sequences $\{t_n\}$ and $\{\epsilon_n\}$, with $0 < \epsilon_n \leq 1$ for every $n$ and $t_n \to \infty$ as $n \to \infty$, such that

$$\left| \int_{t_n}^{t_n+\epsilon_n} h(t) \, dt \right| \geq \alpha$$

(4.13)

for every $n$. Assume (we shall contradict this assumption) that there exist some $t_0 \geq 0$ and some $x_0$ for which the solution $F(t, t_0, x_0)$ of $(P_2)$ satisfies $\|F(t_0, x_0)\| \to 0$ as $t \to \infty$. Choose $T$ so large that $t \geq T$ implies

$$|F(t, t_0, x_0)| < \alpha[3L]^{-1},$$

where $L$ is the Lipschitz constant for $f$ and we may assume without loss of generality that $L \geq 1$. Then for $n$ so large that $t_n \geq T$, we have

$$\left| \int_{t_n}^{t_n+\epsilon_n} h(t) \, dt \right|$$

$$= \left| \int_{t_n}^{t_n+\epsilon_n} f(t, F(t, t_0, x_0)) \, dt + \int_{t_n}^{t_n+\epsilon_n} F'(t, t_0, x_0) \, dt \right|$$

$$\leq \int_{t_n}^{t_n+\epsilon_n} L |F(t, t_0, x_0)| \, dt + |F(t_n + \epsilon_n, t_0, x_0)| + |F(t_n, t_0, x_0)|$$

$$\leq \frac{\alpha}{3} + \frac{\alpha}{3L} + \frac{\alpha}{3L} < \alpha,$$

a contradiction to (4.13). Thus no solution of $(P_2)$ tends to zero as $t \to \infty$ hence the conclusion of Theorem 4.6 certainly does not hold, completing the proof.

5. Perturbed Nonlinear Systems: A Second Proof

Theorem 5.1. Let $x = 0$ be U.A.S. and $f \in C_0$ on $D_M$ for $(N)$. Let $g$ satisfy $(H_1)$ and $h$ satisfy $(H_3)$. Then there exist $T_0 \geq 0$ and $\delta_0 > 0$ such that if $t_0 \geq T_0$ and $|x_0| < \delta_0$, the solution $F(t, t_0, x_0)$ of $(P_1)$ satisfies $|F(t, t_0, x_0)| \to 0$ as $t \to \infty$. In particular, if $g(t, 0) = 0$ and $h(t) \equiv 0$, then $x = 0$ is U.A.S. for $(P_1)$

This result is not as general as Theorem 4.6 because here we need $f \in C_0$ even for the case $h(t) \equiv 0$ (see Example 4.4), and also because we assume $g$ satisfies $(H_1)$ rather than $(H_2)$. However, the proof does not require Lyapunov functions, which is rather curious. Furthermore, almost no extra effort is
needed to handle the case where \( h(t) \neq 0 \), in contrast to the changes needed in Section 4.

**Proof.** Solutions and constants corresponding to the system \((N)\) shall be starred, those for \((P_1)\) shall not. Let \( Q(t) = \sup \{ G(s) : t - 1 \leq s < \infty \} \) on \([1, \infty)\). Then \( Q(t) \searrow 0 \) as \( t \to \infty \). By Lemma 3.4,

\[
\int_{t_0}^{t} \gamma(s) \, ds \leq \int_{t_0-1}^{t} G(s) \, ds \leq Q(t_0)(t - t_0 + 1)
\]

if \( t \geq t_0 \geq 1 \). Also

\[
\left| \int_{t_0}^{t} h(s) \, ds \right| \leq \left| \int_{t_0}^{t_0+1} h(s) \, ds \right| + \cdots + \left| \int_{t_0+m}^{t} h(s) \, ds \right| \\
\leq H(t_0) + \cdots + H(t_0 + m) \\
\leq H(t_0)(t - t_0 + 1).
\]

Let \( B(t) = Q(t) + H(t) \) and let \( L \) be the Lipschitz constant for \( f \). Let \( t_0 \geq 1 \) and \( |x_0| \leq r \). Then if \( |F(t, t_0, x_0)| \leq r \) on \([t_0, t_0 + T]\) for some \( T > 0 \), we have

\[
|F(t, t_0, x_0) - F^*(t, t_0, x_0)| \\
= |x_0 + \int_{t_0}^{t} \left[ f(s, F(s, t_0, x_0)) + g(s, F(s, t_0, x_0)) + h(s) \right] \, ds - x_0 - \int_{t_0}^{t} f(s, F^*(s, t_0, x_0)) \, ds| \\
\leq \int_{t_0}^{t} L |F(s) - F^*(s)| \, ds + \int_{t_0}^{t} \gamma(s) \, ds + \int_{t_0}^{t} h(s) \, ds \\
\leq \int_{t_0}^{t} L |F(s) - F^*(s)| \, ds + B(t_0)(t - t_0 + 1) \\
= B(t_0) + \int_{t_0}^{t} [L |F(s) - F^*(s)| + B(t_0)] \, ds.
\]

From Lemma 3.7,

\[
|F(t, t_0, x_0) - F^*(t, t_0, x_0)| \leq B(t_0)e^{L(t-t_0)} + \int_{t_0}^{t} B(t_0)e^{L(t-s)} \, ds \\
\leq R(t_0)e^{Lr} + R(t_0)e^{Lr}(t - t_0) \\
\leq e^{Lr}(1 + r)B(t_0).
\]

We may assume without loss of generality that \( r \leq \delta_0^* \).
Let $0 < \epsilon \leq r$. Choose $\delta = \delta(\epsilon) = \delta^*(\epsilon/2)$ so that $0 < \delta < \epsilon$. Choose $\tau = \tau(\epsilon) = \tau^*(\delta/2)$. Choose $T_1 = T_1(\epsilon) > 1$ so large that

$$B(T_1) < \delta[e^{L\tau}(1 + \tau)2]^{-1}.$$ 

Let $t_0 \geq T_1$ and $|x_0| < \delta$. Then for as long as $|F(t, t_0, x_0)| \leq \epsilon$ in the interval $[t_0, t_0 + \tau]$, 

$$|F(t, t_0, x_0)| \leq |F(t, t_0, x_0) - F^*(t, t_0, x_0)| + |F^*(t, t_0, x_0)|$$

$$\leq e^{L\tau}(1 + \tau)B(t_0) + \epsilon/2$$

$$\leq e^{L\tau}(1 + \tau)B(T_1) + \epsilon/2$$

$$< \delta/2 + \epsilon/2 \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon \leq r$, $F(t, t_0, x_0)$ can be continued to $[t_0, t_0 + \tau]$ on which $|F(t, t_0, x_0)| < \epsilon$. Let $x_1 = F(t_0 + \tau, t_0, x_0)$. Then

$$|x_1| \leq |x_1 - F^*(t_0 + \tau, t_0, x_0)| + |F^*(t_0 + \tau, t_0, x_0)|$$

$$< \delta/2 + |F^*(t_0 + \tau, t_0, x_0)|$$

$$< \delta/2 + \delta/2 = \delta.$$

Now let $m$ be a positive integer and assume that $|F(t, t_0, x_0)| < \epsilon$ on $[t_0, t_0 + m\tau]$ and $|F(t_0 + m\tau, t_0, x_0)| < \delta$. Let $x_m = F(t_0 + m\tau, t_0, x_0)$. Then for as long as $|F(t, t_0 + m\tau, x_m)| \leq \epsilon$ on the interval $[T_0 + m\tau, t_0 + (m + 1)\tau]$, we have

$$|F(t, t_0 + m\tau, x_m)| \leq |F(t, t_0 + m\tau, x_0) - F^*(t, t_0 + m\tau, x_0)|$$

$$+ |F^*(t, t_0 + m\tau, x_m)|$$

$$\leq e^{L\tau}(1 + \tau)B(t_0 + m\tau) + \epsilon/2$$

$$< \delta/2 + \epsilon/2 \leq \epsilon.$$

Since $\epsilon \leq r$, $F(t, t_0, x_0)$ can be continued to the entire interval $[t_0 + m\tau, t_0 + (m + 1)\tau]$ on which $|F(t, t_0, x_0)| < \epsilon$. Let $x_{m+1} = F(t_0 + (m + 1)\tau, t_0, x_0)$. Then

$$|x_{m+1}| \leq |F(t_0 + (m + 1)\tau, t_0 + m\tau, x_m) - F^*(t_0 + (m + 1)\tau, t_0 + m\tau, x_m)|$$

$$+ |F^*(t_0 + (m + 1)\tau, t_0 + m\tau, x_m)|$$

$$< \delta/2 + \delta/2 = \delta.$$

Thus, by induction, $|F(t, t_0, x_0)| < \epsilon$ on every interval $[t_0 + m\tau, t_0 + (m + 1)\tau]$, and hence on $[t_0, \infty)$. Hence if $g(t, 0) = 0$ and $h(t) = 0$, then we have shown that $x = 0$ is uniformly stable. For the rest of the proof, choose $\delta_0 = \delta(r)$ and $T_0 = T_1(r)$. Fix $t_0 \geq T_0$ and $|x_0| < \delta_0$. Then $|F(t, t_0, x_0)| < \epsilon$ on $[t_0, \infty)$. 

Let $0 < \eta < r$. Choose $\delta(\eta) = \delta^*(\eta/2)$, $0 < \delta < \eta$, $\tau(\eta) = T^*(\delta/2)$, and $T_1(\eta)$ so that

$$B(T_1) < \delta[e^{L_\tau(1 + r)}/2]^{-1}.$$ 

Let $y_0 = F(t_0 + T_1, t_0, x_0)$. Then $|y_0| < r \leq \delta^*_0$. Then

$$|F(t_0 + \tau + T_1, t_0 + T_1, y_0)| \leq |F(t_0 + \tau + T_1, t_0 + T_1, y_0) - F^*(t_0 + \tau + T_1, t_0 + T_1, y_0)|$$
$$+ |F^*(t_0 + \tau + T_1, t_0 + T_1, y_0)|$$
$$\leq e^{L_\tau(1 + r)}B(t_0 + T_1) + \delta/2$$
$$< \delta/2 + \delta/2 = \delta.$$ 

Thus by the first part of the proof,

$$|F(t, t_0, x_0)| = |F(t, t_0 + \tau + T_1, F(t_0 + \tau + T_1, t_0, x_0))| < \eta$$

for all $t \geq t_0 + T$, where $T = T(\eta) = \tau + T_1$, completing the proof.

References